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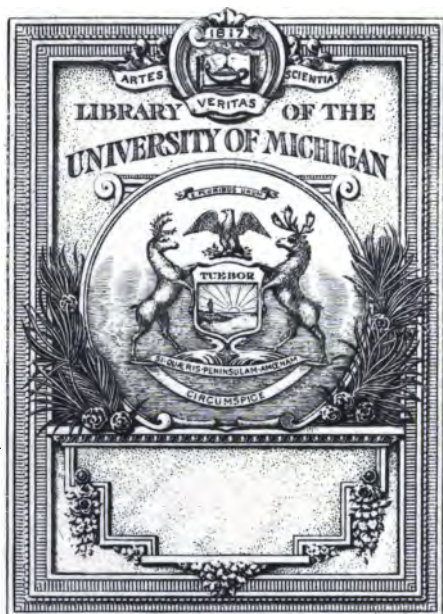
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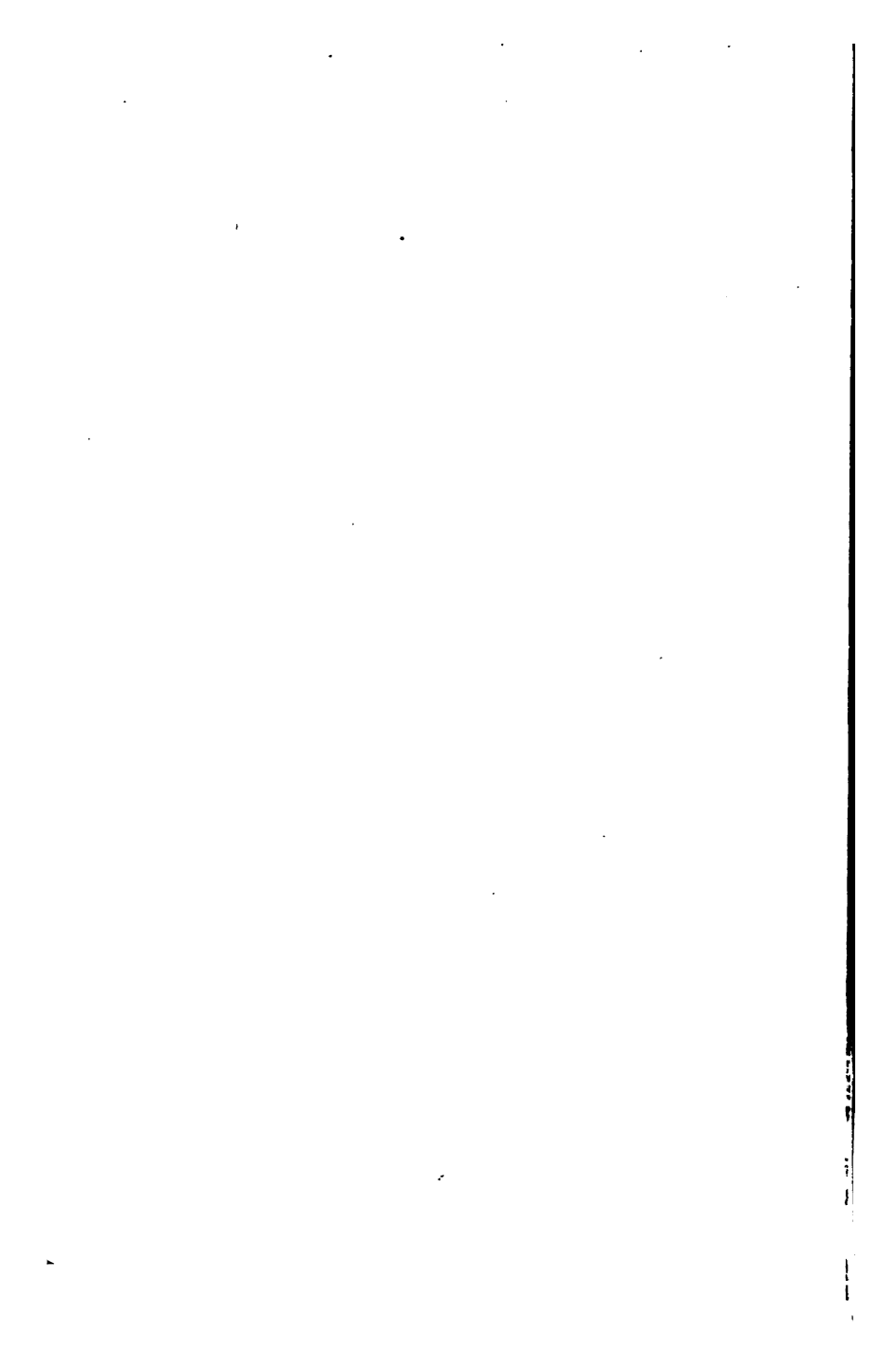


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1891



THE ELEMENTS
OF
PLANE AND SOLID GEOMETRY.

WITH NUMEROUS EXERCISES.

BY
EDWARD A. BOWSER, LL.D.,
PROFESSOR OF MATHEMATICS AND ENGINEERING IN RUTGERS COLLEGE.

SECOND EDITION.

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PREFACE.

IN the present work on Elementary Geometry, or what is commonly known as the Euclidian Geometry, it is aimed to combine the excellencies of Euclid with those of the best modern writers, especially of Legendre, and Rouché and Comberousse. Many of the demonstrations are those of Euclid, with minor changes frequently introduced, and the syllogistic form is retained throughout ; but the arrangement is quite different.

Many objections have been made against Euclid. His definitions are not all of them the best, nor are they in their proper places. His treatment of angles is deficient. His arrangement of the propositions is often poor, mixing straight lines, angles, triangles, etc., without any regular classification ; and his demonstrations are sometimes cumbersome and prolix.

Nevertheless, although numerous attempts have been made to improve upon Euclid, it still remains the great model, the unrivalled original, on which is founded the whole system of elementary Geometry. Perhaps a more finished specimen of exact logic has never been produced than that of the old Greek Geometer.

In the present treatise it is desired to effect two objects : (1) to teach geometric truths ; (2) to discipline and invigorate the mind, to train it to habits of clear and consecutive reasoning. Accordingly, more numerous propositions have been given, and the demonstrations made more complete, than either object alone would seem to demand.

In each proposition is a distinct statement, of what is given, of what is required, and of the proof. Each assertion in the proof begins a new line, and is accompanied by a reference to the preceding principle on which the assertion depends. These references are quoted two or three times in small type, and afterwards referred to only by number. *The student should always be ready, if required, to quote the proper reference, and to show its application.* The text is so arranged that the enunciation, figure, and proof of each proposition

are in view together ; and notes are directly appended to the propositions to which they refer.

A few symbols and abbreviations of words have been freely used, but only such as have long been employed by mathematicians, and are recognized by the majority of teachers.

In the figures the *given* lines are represented by *full* lines, those which are added to *aid* in the demonstration, by *short-dotted* lines, and the *resulting* lines by *dashed*, or *long-dotted* lines. In solid geometry the dotted lines represent those which a solid body would conceal.

The propositions marked with a star may be omitted without interfering with the continuity of the work, but the omission is not recommended.

The exercises distributed through the text are quite easy, and may all be worked out by the average student ; those at the end of each book are more advanced, and have been carefully graded, with hints appended to many of the more difficult ones.

It is only in original demonstration that the student can acquire mental power. More discipline is gained in working out one demonstration, without aid, than by learning a number of them that are given by others. A student can never really *comprehend* a subject if he only tries to understand and remember what the book says. The subject can become *known* to him only by his *thinking* upon it. To develop the power of independent thought in the student, is the most important part of the teacher's work, and it is the most difficult.

In preparing this work I have consulted some of the best American and English books on Euclidian Geometry, and am especially indebted to the text-book recently published by the English Association for the Improvement of Geometrical Teaching. I have also derived assistance from a number of French works, especially from those of Catalan, Briot, and Legendre, while the *Traité de Géométrie* by Rouché and De Comberousse has been my constant companion.

It remains for me to express my thanks to Prof. R. W. Prentiss, of the Nautical Almanac Office, for reading the MS., and to Mr. I. S. Upson, the College Librarian, for reading the proof-sheets.

E. A. B.

RUTGERS COLLEGE,
NEW BRUNSWICK, N. J., May, 1890. }

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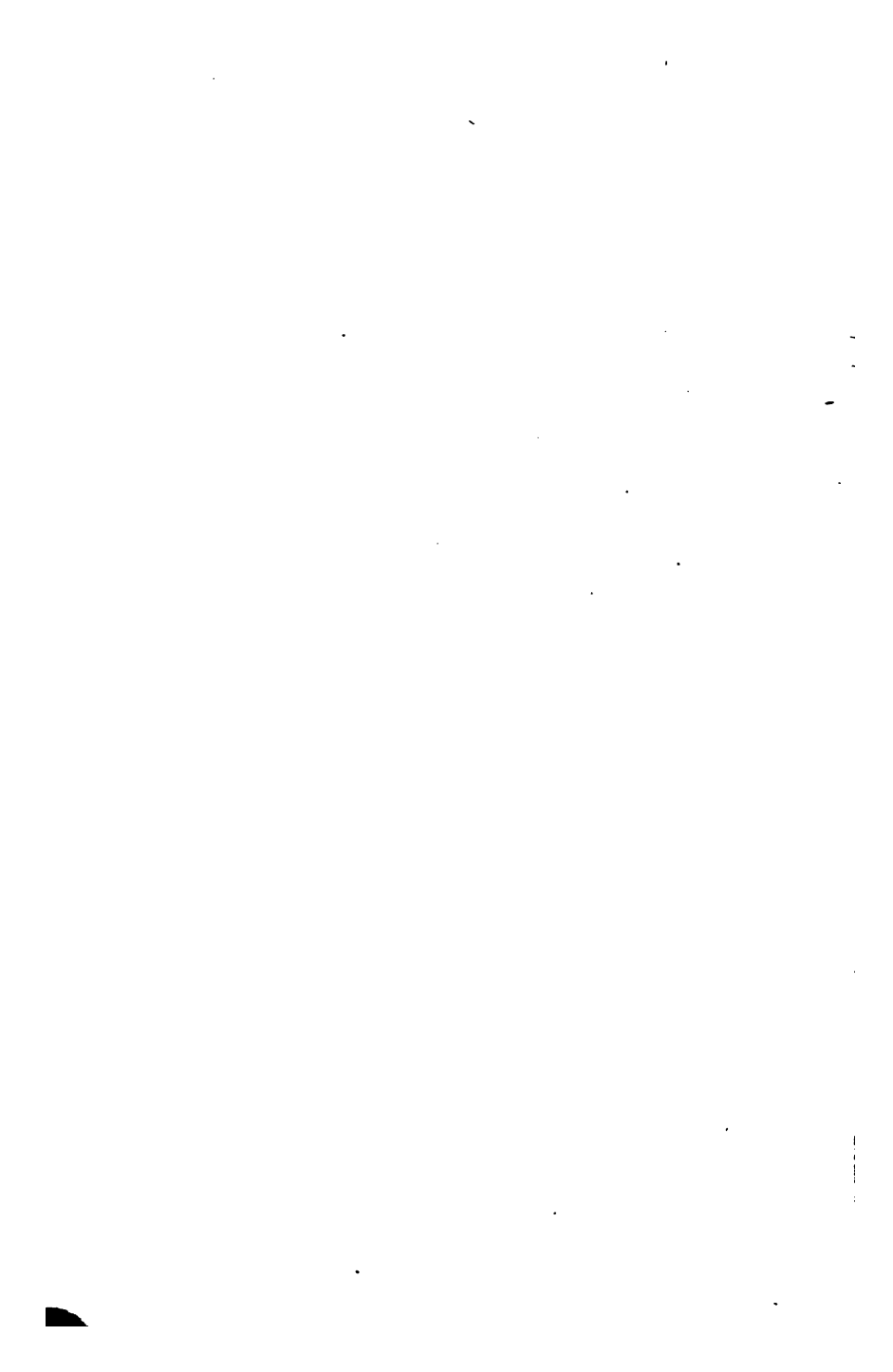
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ELEMENTARY GEOMETRY.

INTRODUCTION.

DEFINITIONS.

1. SPACE is indefinite extension in every direction. All material bodies occupy limited portions of space, and have *length, breadth, thickness, form, and position*. The material body occupying any portion of space is called a *physical solid*. The part of space which is or may be occupied by a material body is called a *geometric solid*. A physical solid is therefore a *real body*, while a geometric solid is only the *form* of a physical solid, and is the one treated of in Geometry. The term *solid* will be used for brevity to denote a geometric solid.

2. A *solid* is a limited portion of space, and has *length, breadth, and thickness*. *Length, breadth, and thickness* are called the *three dimensions* of the solid.

3. A *surface* is the limit or boundary of a solid, and has only two dimensions, *length* and *breadth*.

A surface has no thickness, for if it had any, however small, it would form part of the solid, and would be space of three dimensions.

4. A *line* is the limit or boundary of a surface, and has only one dimension, namely, *length*.

A line has no breadth, for if it had any, however small, it would form part of the surface, and would be space of two dimensions; and if in addition it had any thickness, it would be space of three dimensions; hence a line has neither breadth nor thickness.

5. A *point* is the limit or extremity of a line, and has *position*, but neither *length*, *breadth*, nor *thickness*.

A point has no length, for if it had any, however small, it would form part of the line of which it is the extremity; and it can have neither breadth nor thickness because the line has none.

6. If we suppose a solid to be divided into two parts which touch each other, the division between the two parts is a surface. This surface can have no thickness, for if it had a thickness, however small, it would be a part either of the one solid or the other, and would therefore be a solid and not a surface.

Again, if we suppose a surface cut into two parts which touch each other, the division between the two parts is a line. This line can have no thickness, because the surface has none, and it can have no breadth, for it forms no part of either surface.

If we suppose a line cut into two parts which touch each other, the division between the two parts is a point. This point can have neither breadth nor thickness, because the line has none, and it can have no length, for it forms no part of either line.

Euclid regarded a point merely as a *mark of position*, and he attached to it no idea of size and shape.

Similarly, he considered that the properties of a line arise only from its *length* and *position*, without reference to that minute breadth which every line must really have if *actually drawn*, even though the most perfect instruments are used.

We cannot make the points, lines, and surfaces of Geometry. A dot, made on paper or on the blackboard, will have length, breadth, and thickness, and hence will not be a real point. Yet the dot may be taken as an imperfect representation of the real point. So also a line, drawn on paper or on the blackboard, will have breadth and thickness, and hence will not be a real line. Yet the line which we draw may be taken to represent the real line.

7. We have considered a surface as the boundary of a solid, a line as the boundary of a surface, and a point as the limit of a line. On the other hand, inversely, we may re-

gard a line as generated by the motion of a point, a surface as generated by the motion of a line, and a solid as generated by the motion of a surface. Again, each of these may be regarded in a purely abstract manner, distinct from each other.

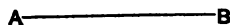
Thus, we may suppose a surface to exist in space separately from the solid whose boundary it forms, and to be of *unlimited extent*.

Similarly, we may suppose a line to exist in space separately from the surface whose boundary it forms, and to be of *unlimited length*.

Likewise we may suppose a point to exist in space separately from the line, and to have only position.

The points, lines, surfaces, and solids of Geometry are called *geometric points, lines, surfaces, and solids*.

8. A *straight line*, or *right line*, is one which has the same direction at every point, as the line AB.



9. A *curved line* is one no part of which is straight, but changes its direction at every point, as the line CD.



10. A *broken line* is a line made up of different successive straight lines, as the line EF.

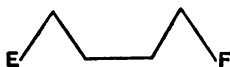


Fig. 1

The word *line*, used alone, signifies a straight line; and the word *curve*, a curved line.

11. A *plane surface*, or, simply, a *plane*, is a surface in which the right line joining any two points in it lies wholly in the surface.

12. A *curved surface* is one no part of which is plane.

13. A *figure* is any definite combination of points, lines, surfaces, or solids.

A *plane figure* is one formed of points and lines in a plane. If the figure is formed of right lines only, it is called a *rectilinear*, or *right-lined*, figure.

The figure of a solid depends upon the relative position

of the points in its surface. Lines, surfaces, and solids are the geometric figures. When the *extent* of lines, surfaces, and solids is considered they are called *magnitudes*, but when their *form* or *shape* is considered they are called *figures*.

14. *Geometry* is the science which treats of magnitude, form, and position. Thus it is the province of Geometry to investigate the properties of solids, of surfaces, and of the figures constructed on surfaces.

Plane Geometry treats of plane figures.

Solid Geometry, called also *Geometry of Space* and *Geometry of Three Dimensions*, treats of solids, of curved surfaces, and of the figures described on curved surfaces.

STRAIGHT LINES.

15. A *finite straight line* is a straight line contained between two definite points which are its extremities. When a straight line is produced indefinitely it is called an *indefinite straight line*. Any finite straight line may be supposed at any time to be produced into an indefinite straight line.

Two finite straight lines are said to be *equal*, or of *equal* length, when the extremities of the one line can be made to coincide respectively with the extremities of the other.

If any line, as OB, be produced through O to A, the parts OB and OA are said to have *opposite directions* from the common point O.

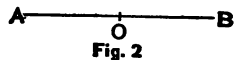


Fig. 2

Every straight line AB has two opposite directions, the one from A toward B, expressed by "the line AB," and the other from B toward A, expressed by "the line BA."

If a line BC is to be produced toward D, we should express this by saying that "BC is to be produced"; but if it is to be produced toward A, we should express this by saying that "CB is to be produced."



Fig. 3

Straight lines are *added* together by placing them one

after another in succession in the same straight line so that one extremity of each newly added line coincides with one extremity of the last added line, and so that no part of any newly added line coincides with any part of the last added line.

Thus, AB, BC, and CD, Fig. 3, are added together and form the straight line AD.

AB, BC, and CD are called the *parts* of AD, and AD is called the *sum* of AB, BC, and CD.

PLANE ANGLES.

16. A *plane angle* is the opening between two straight lines drawn from the same point. The straight lines are called the *arms* or *sides* of the angle, and the common point is called its *vertex*. Thus the lines OA, OB are said to *contain*, or *include*, or *form* the angle at O.

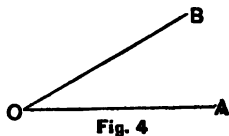


Fig. 4

When there are several angles at one point, any one of them is expressed by three letters, putting the letter at the vertex between the other two. Thus, if the straight lines OA, OB, OC meet at the point O, the angle contained by the lines OA, OB is named the angle AOB or BOA; the angle contained by the lines OA, OC is named the angle AOC or COA; and the angle contained by OB, OC is named the angle BOC or COB.

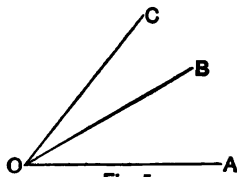


Fig. 5

When there is only one angle at a point, it may be denoted either by the single letter at that point, or by three letters as above. Thus in Fig. 4 the angle at the point O may be denoted either by the angle O or by AOB or by BOA.

17. *Adjacent angles* are angles which have a common vertex and one common arm, their non-coincident arms

being on opposite sides of the common arm. Thus the angles AOB and COB (Fig. 5) are adjacent angles, of which OB is the common arm.

Of the two straight lines OB, OC (Fig. 5) it is easily seen that the *opening* between OA and OC is greater than the opening between OA and OB. This we express by saying that the angle AOC is greater than the angle AOB.

The *magnitude* or *size* of an angle depends entirely upon the extent of opening between its sides, and is not altered by changing the length of its sides.

18. Angles are *equal* when they can be placed one upon the other so that the vertex and sides of the one can be made to coincide with the vertex and sides of the other. Thus the angles ABC and DEF are equal if ABC can be placed upon DEF so that while BA coincides with ED, BC shall also coincide with EF.

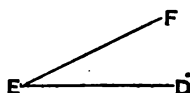
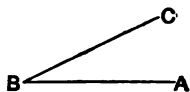


Fig. 6

19. The angle formed by joining two or more angles together is called their *sum*. Angles are *added together* by placing them so as to be adjacent to each other. Thus the sum of the two angles ABC, PQR, is the angle ABR, formed by applying the side QP to the side BC so that the vertex Q shall fall on the vertex B, and the side QR on the opposite side of BC from BA.

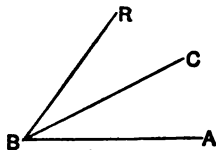
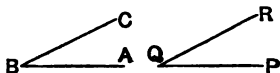


Fig. 7

If the angles ABC, PQR are equal to each other, the angle ABR is *double* either of them, and the common side BC is said to *bisect* the angle between the non-coincident sides BA and BR.

20. When a straight line standing on another makes the adjacent angles equal to each other, each of the angles is

called a *right angle*; and the straight line which stands on the other is said to be *perpendicular* or *at right angles* to it. Thus, if the adjacent angles AOC and BOC are equal to each other, each is a right angle, and the line CO is perpendicular to AB. The point O is called the *foot* of the perpendicular.

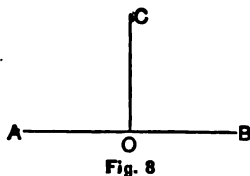


Fig. 8

21. A *straight angle* has its arms extending in opposite directions so as to be in the same straight line. Thus, if the arms OA, OB are in the same straight line, the angle formed by them is called a straight angle.

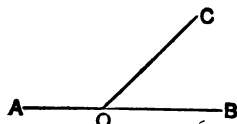


Fig. 9

Since the sum of the two right angles AOC and BOC (Fig. 8) is the angle AOB (19)*, a right angle is half a straight angle.

22. An *acute angle* is an angle which is less than a right angle, as the angle A.

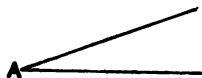


Fig. 10

23. An *obtuse angle* is an angle which is greater than a right angle, and less than a straight angle, as the angle BAC.

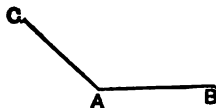


Fig. 11

24. When the sum of two angles is a right angle, each is called the *complement* of the other, and the two are called *complementary angles*. Thus, if the angle BAC is a right angle, the angles BAD, DAC are complements of each other.

25. When the sum of two angles is a straight angle, each is called the *supplement* of the other, and the two are called *supplementary adjacent angles*. Thus, if the angle AOB (Fig. 9) is a straight angle, the angles BOC, COA are supplements of each other.

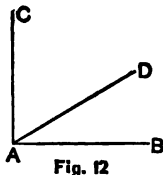


Fig. 12

* An Arabic numeral in parenthesis refers to an *article*.

Hence, when one line stands on another, the two adjacent angles are supplements of each other. Hence a right angle is equal to its supplement.

The supplement of an acute angle is obtuse, and, conversely, the supplement of an obtuse angle is acute.

26. A *reflex angle* is an angle which is greater than a straight angle, and less than two straight angles, as the angle O.

Acute, obtuse, and reflex angles are called *oblique* angles, in distinction from right and straight angles; and intersecting lines which are not perpendicular to each other are called *oblique* lines.

27. Where two angles are contained between two intersecting lines on opposite sides of the vertex, they are called *opposite* or *vertical* angles. Thus, AOC and BOD are opposite or vertical angles, as also AOD and COB.

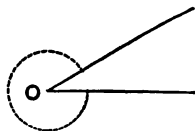


Fig. 13

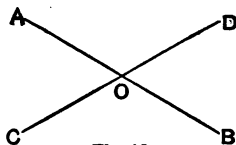


Fig. 14

ANGULAR MEASURE.

28. A right line drawn from the vertex and turning about it in the plane of the angle from the position of coincidence with one side of the angle to that of coincidence with the other side, is said to turn through the angle, and the angle is the greater as the quantity of turning is greater.

Thus, suppose that the right line OP (Fig. 15) is capable of revolving about the point O, like the hands of a watch, but in the opposite direction, and that it has passed successively from the position OA to the positions occupied by OB, OC, OE, etc. Then it is clear that the line *must have done more turning* in passing from OA to OC than in passing

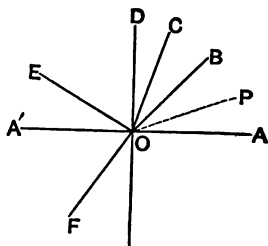


Fig. 15

from OA to OB; and consequently the angle AOB is said to be greater than the angle AOB.

When the revolving line OP turns from coincidence with OA to the position OB it is said to *describe* or to *generate* the angle AOB. When the revolving line has turned from the position OA to the position OD, perpendicular to OA, it has generated the right angle AOD; when it has turned to the position OA', it has generated the straight angle AOA'; when it has turned to the position OF, it has generated the reflex angle AOF; when it has turned entirely around to the position OA, it has generated two straight angles.

Hence, the whole angle which a line must turn through, about a point in a plane, to take it around to its first position, is two straight angles, or four right angles.

Again, since the revolving line may turn from one position to the other in either of two directions, two angles are formed by two lines drawn from a point.

Thus, if OA, OB be the sides of an angle, a line may turn from the position OA to the position OB about the point O in *either* of the two directions indicated by the arrows, giving the obtuse angle AOB (marked *a*), and the reflex angle AOB (marked *b*).

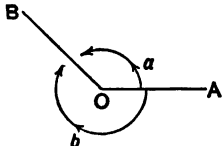


Fig. 16

Angles are generally measured in *degrees, minutes, and seconds*. A *degree* is the *ninetieth part* of a right angle, or the *three hundred and sixtieth part* of four right angles. A *minute* is the *sixtieth part* of a degree; and a *second* is the *sixtieth part* of a minute. Degrees, minutes, and seconds are denoted by the symbols $^{\circ}$, $'$, $''$. Thus 7 degrees, 24 minutes, and 38 seconds is written, $7^{\circ} 24' 38''$.

Hence, when the revolving line OP (Fig. 15) has turned through *one-fourth* of a revolution, it has generated a right angle, or 90° ; when it has made *half* a revolution, it has generated a straight angle, or 180° ; and when it has made

a *whole* revolution, it has generated two straight angles, or 360° .

SUPERPOSITION.

29. The placing of one geometric magnitude on another, such as a line on a line, an angle on an angle, etc., is called *superposition*. The superposition employed in Geometry is only *mental*, that is, we conceive one magnitude to be taken up and laid down upon the other; and then, if we can prove that they coincide, we infer that they are equal. This is the ultimate test of the equality of two geometric magnitudes.

Thus, if two straight lines are to be compared, we conceive one to be taken up and placed on the other, and find whether their two ends can be made to coincide. If so, they are equal; if not, they are unequal. If two angles are to be compared, we conceive one to be taken up and applied to the other. If they can be so placed that their vertices coincide in position and their sides in direction, the angles are equal (18).

Superposition involves the principle* that "any figure may be taken up, transferred from one position to another, and laid down again without change of form or size."

Magnitudes which coincide with one another throughout their whole extent are said to be *equal*.

DEFINITIONS OF TERMS.

30. A *theorem* is a truth which requires proof.

31. A *problem* is a question which requires solution, such as a particular line to be drawn, or a required figure to be constructed.

32. An *axiom* is a self-evident truth, which is admitted without proof.

33. A *postulate* assumes the possibility of solving a certain problem.

* Euclid makes frequent use of this principle, without explicitly stating it.—CASEY.

34. A *proposition* is a general term for a theorem, problem, axiom, or postulate.

35. A *demonstration* is the course of reasoning by which we prove a theorem to be true.

36. A *corollary* is a conclusion which follows immediately from a theorem.

37. A *lemma* is an auxiliary theorem required in the demonstration of a principal theorem.

38. A *scholium* is a remark upon one or more propositions.

39. An *hypothesis* is a supposition made either in the enunciation of a proposition or in the course of a demonstration.

40. A *solution* of a problem is the method of construction which accomplishes the required end.

41. A *construction* is the drawing of such lines and curves as may be required to prove the truth of a theorem, or to solve a problem.

42. The *enunciation* of a theorem consists of two parts: the *hypothesis*, or that which is assumed; and the *conclusion*, or that which is asserted to follow therefrom.

Thus, in the typical theorem,

If A is B, then C is D,

the hypothesis is that A is B, and the conclusion that C is D.

43. The *enunciation* of a problem consists of two parts: the *data*, or things supposed to be given; and the *quæsitæ*, or things required to be done.

44. Two theorems are said to be *converse*, each of the other, when the hypothesis of each is the conclusion of the other.

Thus, If C is D, then A is B,

is the converse of the typical theorem in (42).

POSTULATES.

45. Let it be granted—

1. That a straight line may be drawn from any one point to any other point.
2. That a terminated straight line may be produced to any distance in a straight line.
3. That a circle may be described with any centre, at any distance from that centre.

AXIOMS.

46. 1. Things which are equal to the same thing are equal to each other.

2. If equals be added to equals the sums will be equal.

3. If equals be taken from equals the remainders will be equal.

4. If equals be added to unequals the sums will be unequal.

5. If equals be taken from unequals the remainders will be unequal.

6. Things which are double the same thing, or equal things, are equal to one another.

7. Things which are halves of the same thing, or of equal things, are equal to one another.

8. The whole is greater than any of its parts.

9. The whole is equal to the sum of all its parts.

GEOMETRIC AXIOMS.

10. A straight line is the shortest distance between any two points.

11. If two straight lines have two points in common, they will coincide throughout their whole length, and form but one straight line.

12. Through a given point only one straight line can be drawn parallel to a given straight line.

SYMBOLS AND ABBREVIATIONS.

47. The following are some of the principal symbols and abbreviations that are commonly used in Colleges and Public Schools. These symbols and abbreviations are generally employed in the present book. They are recommended to the student for writing out the propositions and demonstrations on the blackboard or in exercise books, as their use will greatly shorten the work.

+	plus, or together with.	adj.	adjacent.
-	minus, or diminished by.	alt.	alternate.
×	multiplied by.	ax.	axiom.
÷	divided by.	cons.	construction.
∴	therefore.	cor.	corollary.
=	{ equals, { or is (or are) equal to.	cyl.	cylinder.
>	is (or are) greater than.	def.	definition.
<	is (or are) less than.	ext.	exterior.
∠	angle.	fig.	figure.
∠s	angles.	hyp.	hypothesis.
Δ	triangle.	int.	interior.
Δs	triangles.	opp.	opposite.
▭	parallelogram.	post.	postulate.
▭s	parallelograms.	prob.	problem.
⊥	perpendicular.	prop.	proposition.
⊥s	perpendiculars.	pt.	point.
	parallel.	rt.	right.
s	parallels.	sim.	similar.
○	circle.	sq.	square.
○s	circles.	st.	straight.
○ _{ce}	circumference.	sup.	supplementary.
○ _{ces}	circumferences.		

The words "join AB" are used as an abbreviation for "draw a straight line from A to B."

The initial letters Q. E. D., placed at the end of a theorem, stand for the Latin words **Quod erat Demonstrandum**, meaning *which was to be proved*.

The letters Q. E. F., placed at the end of a problem, stand for **Quod erat Faciendum**, *which was to be done*.

PLANE GEOMETRY.

BOOK I.

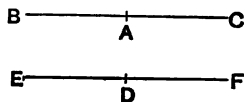
RECTILINEAR FIGURES.

PERPENDICULAR AND OBLIQUE LINES.

Proposition 1. Theorem.

48. *All straight angles are equal to one another.**

Hyp. Let AB, AC be the arms of a st. $\angle$ whose vertex is A, and DE, DF the arms of another st. $\angle$ whose vertex is D.



To prove $\angle BAC = \angle EDF$.

Proof. Because the $\angle BAC$ is a st. $\angle$,

$\therefore$ BA and AC are in the same st. line BAC. (21)

Because the $\angle EDF$ is a st. $\angle$,

$\therefore$ ED and DF are in the same st. line EDF. (21)

Now if the $\angle BAC$ be applied to the $\angle EDF$ so that the vertex A shall coincide with the vertex D and the arm AB with the arm DE, then the arm AC will coincide with the arm DF,

because, if two st. lines have two points in common they will coincide throughout their whole length, and form but one st. line. (Ax. 11)

$\therefore \angle BAC = \angle EDF$. Q.E.D.

49. COR. 1. *All right angles are equal to one another.†* (21), also (Ax. 7).

50. COR. 2. *The complements of equal angles are equal to each other, and the supplements of equal angles are equal to each other.*

51. COR. 3. *At a given point in a given straight line only one perpendicular can be drawn to that line.*

* The Elements of Plane Geometry, by Association for the Improvement of Geometric Teaching, p. 20a.

† This is Euclid's 11th axiom.

Proposition 2. Theorem.

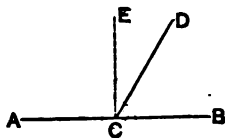
52. *If one straight line meet another straight line, the sum of the two adjacent angles is equal to two right angles.*

Hyp. Let the st. line DC meet the st. line AB at C.

To prove

$$\angle ACD + \angle DCB = 2 \text{ rt. } \angle \text{s.}$$

Proof. From C draw CE at rt. $\angle$ s to AB.



Then $\angle ACD = \angle ACE + \angle ECD.$

Add $\angle DCB$ to each.

$$\therefore \angle ACD + \angle DCB = \angle ACE + \angle ECD + \angle DCB.$$

If equals be added to equals, the sums will be equal (Ax. 2).

Again, $\angle ECB = \angle ECD + \angle DCB.$

Add $\angle ACE$ to each.

$$\therefore \angle ACE + \angle ECB = \angle ACE + \angle ECD + \angle DCB.$$

If equals be added to equals, the sums will be equal (Ax. 2).

Hence

$$\angle ACE + \angle ECB = \angle ACD + \angle DCB. \quad (\text{Ax. 1})^*$$

But $\angle ACE + \angle ECB = 2 \text{ rt. } \angle \text{s.} \quad (\text{Cons.})$

$$\therefore \angle ACD + \angle DCB = 2 \text{ rt. } \angle \text{s.} \quad \text{Q. E. D.}$$

SCH. *The angles ACD, DCB are supplementary adjacent angles.* (25)

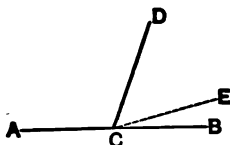
53. COR. *If one of the angles ACD, DCB is a right angle, the other is also a right angle.*

The sum of all the angles on the same side of a straight line, at a common point, is equal to two right angles; for their sum is equal to the sum of the two adjacent angles ACD, DCB.

* The student should quote every reference in full.

Proposition 3. Theorem.

54. *Conversely, if the sum of two adjacent angles is equal to two right angles, their exterior sides are in one straight line.*



Hyp. Let $\angle ACD + \angle DCB = 2 \text{ rt. } \angle s.$

To prove that AC and CB are in one st. line.

Proof. If CB be not in the same st. line as AC, let CE be in the same st. line as AC.

Then $\angle ACD + \angle DCE = 2 \text{ rt. } \angle s,$

being sup. adj. } \angle s (52).

But $\angle ACD + \angle DCB = 2 \text{ rt. } \angle s. \quad (\text{Hyp.})$

$\therefore \angle ACD + \angle DCE = \angle ACD + \angle DCB. \quad (\text{Ax. 1})$

Take away the common $\angle ACD.$

$\therefore \angle DCE = \angle DCB, \quad (\text{Ax. 3})$

which is impossible (Ax. 8), unless CE coincides with CB.

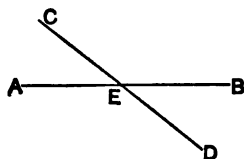
$\therefore$ AC and CB are in one st. line. Q. E. D.

EXERCISES.

1. Find the number of degrees in an angle if it is three times its complement.
2. Find the number of degrees in an angle if its complement and supplement are together equal to 110° .

Proposition 4. Theorem.

55. *If two straight lines intersect each other, the vertical angles are equal.*



Hyp. Let the st. lines AB and CD cut at E.

To prove $\angle AEC = \angle BED$.

Proof. $\angle CEA + \angle CEB = 2 \text{ rt. } \angle s$,

being sup. adj. $\angle s$ (52).

$\angle AEC + \angle AED = 2 \text{ rt. } \angle s$,

being sup. adj. $\angle s$ (52).

$\therefore \angle CEA + \angle CEB = \angle AEC + \angle AED$.

All rt. $\angle s$ are equal to one another (49).

Take away the common $\angle AEC$.

$\therefore \angle CEB = \angle AED$. (Ax. 3)

In the same way it may be proved that

$\angle AEC = \angle BED$. Q.E.D.

56. COR. 1. *If two straight lines intersect each other, the four angles which they make at the point of intersection, are together equal to four right angles.*

If one of the four angles is a right angle, the other three are right angles, and the lines are mutually perpendicular to each other.

57. COR. 2. *If any number of straight lines meet at a point, the sum of all the angles made by consecutive lines, in a plane, is equal to four right angles.*

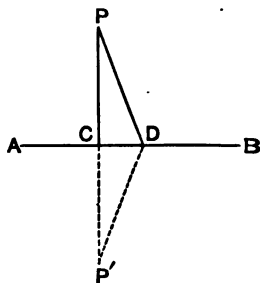
Proposition 5. Theorem.

58. *From a point without a straight line only one perpendicular can be drawn to that line; and this perpendicular is the shortest distance from the point to the line.*

Hyp. Let AB be the given st. line, P the point without the line, PC a $\perp$ from P to AB , and PD any other line from P to AB .

(1) *To prove* that PD is not $\perp$ to AB .

Proof. Let the part of the plane above the line AB , and which contains P , be revolved about AB as an axis until the point P comes into the position P' .



Then, since $\angle ACP$ is a rt. $\angle$, $\angle ACP'$ is also a rt. $\angle$. (29)

$\therefore PCP'$ is a st. line. (54)

If $\angle ADP$ is a rt. $\angle$, $\angle ADP'$ is also a rt. $\angle$. (29)

$\therefore PDP'$ is a st. line; (54)

that is, between two points there are two straight lines, which is impossible. (Ax. 11)

$\therefore \angle ADP$ is not a rt. $\angle$, and $\therefore PD$ is not $\perp$ to AB . (20)

$\therefore PC$ is the only $\perp$ to AB from the point P .

(2) *To prove* $PC < PD$.

Proof. Since in the revolution of CPD about AB as an axis, the point P comes into the position P' ,

$\therefore PC = P'C$, and $PD = P'D$; (29)

$\therefore PC + P'C = 2 PC$, (Ax. 2)

and $PD + DP' = 2 PD$. (Ax. 2)

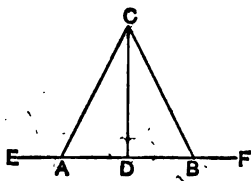
But $PC + P'C < PD + DP'$. (Ax. 10)

$\therefore 2PC < 2PD$, and $\therefore PC < PD$. Q. E. D.

59. SCH. By the distance of a point from a line is meant the *shortest distance*, that is, the length of the perpendicular from the point to the line.

Proposition 6. Theorem.

60. *Two oblique lines drawn from a point to a straight line, cutting off equal distances from the foot of the perpendicular, are equal.*



Hyp. Let CD be the $\perp$ from C to the line EF , and CA and CB two oblique lines so that $AD = DB$.

To prove $CA = CB$.

Proof. Let the part CAD be revolved about CD as an axis until it comes into the plane of CDB .

Because $\angle CDA = \angle CDB = \text{rt. } \angle$, (Hyp.)

$\therefore DE$ will take the direction of DF .

Because $DA = DB$, (Hyp.)

the point A will fall upon the point B .

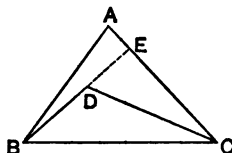
$\therefore CA = CB$. (Ax. 11)

Q. E. D.

61. COR. *Two equal oblique lines drawn from a point to a straight line, make equal angles with that line, and cut off equal distances from the foot of the perpendicular.*

Proposition 7. Theorem.

62. *If from any point two lines be drawn to the ends of a straight line, their sum will be greater than the sum of two other lines similarly drawn, but included by them.*



Hyp. Let AB, AC be two lines drawn from the point A to the ends of the line BC, and DB, DC two lines similarly drawn, but included by AB, AC.

To prove $AB + AC > DB + DC$.

Proof. Produce BD to meet AC at E.

Then $BA + AE > BE$. (Ax. 10)

Add EC to each.

$\therefore BA + AC > BE + EC$. (Ax. 2)

Again, $DE + EC > DC$. (Ax. 10)

Add BD to each.

$\therefore BE + EC > BD + DC$. (Ax. 2)

Much more $\therefore BA + AC > BD + DC$. Q.E.D.

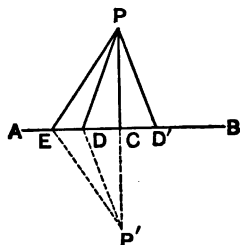
EXERCISES.

1. Write out in full the proof of the second part of Prop. 4, that the angle AEC is equal to the angle BED.

2. Prove that the straight line which bisects the angle AEC, in Prop. 4, bisects also the vertical angle BED.

Proposition 8. Theorem.

63. *Of two oblique lines drawn from the same point to the same straight line, that which meets the line at the greater distance from the foot of the perpendicular is the greater.*



Hyp. Let PC be $\perp$ from P to AB, and PE, PD two oblique lines so that $CE > CD$.

To prove $PE > PD$.

Proof. Produce PC to P', making $CP' = PC$.

Join DP', EP'.

Because AB is $\perp$ to PP' at its middle point,

$$\therefore PD = DP', \text{ and } PE = EP'. \quad (60)$$

$$\text{But } PE + EP' > PD + DP'. \quad (62)$$

$$\therefore 2PE > 2PD, \therefore PE > PD.$$

If the two oblique lines are on opposite sides of PC, as PE and PD', and if $CE > CD'$, take $CD = CD'$, and join PD.

$$\text{Then } PD = PD'. \quad (60)$$

$$\text{But } PE > PD, \text{ as just proved.}$$

$$\therefore PE > PD'. \quad \text{Q. E. D.}$$

64. COR. 1. *Only two equal straight lines can be drawn from a point to a straight line.*

65. COR. 2. *Of two unequal oblique lines, the greater cuts off the greater distance from the foot of the perpendicular.*

Proposition 9. Theorem.

66. 1. *Every point in the perpendicular erected at the middle of a straight line is equally distant from the extremities of the line.*

2. *Every point without the perpendicular is unequally distant from the extremities of the line.*

Hyp. Let DC be $\perp$ to AB at its middle point C , P any point in DC , and O any point without DC .

Draw PA , PB , and OA , OB .

(1) *To prove* $PA = PB$.

Proof. Because $DC \perp$ to AB , and $AC = CB$,

$$\therefore PA = PB. \quad (60)$$

(2) *To prove* $OA > OB$.

Proof. OA will cut the $\perp$ DC at E ; join EB .

Then $OB < OE + EB$. (Ax. 10)

But $EB = EA$. (60)

$$\therefore OB < OE + EA,$$

or $OB < OA$. Q. E. D.

67. COR. *Every point equally distant from the extremities of a straight line lies in the perpendicular bisector of the line.*

If a straight line have two points, each of which is equally distant from the extremities of a second line, it will be perpendicular to the second line at its middle point.

EXERCISES.

1. Prove that the bisectors of two vertical angles are in the same straight line.

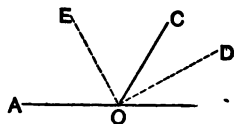
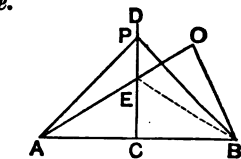
2. Prove that the bisectors of two supplementary adjacent angles are perpendicular to each other.

NOTE.—In the adjoining figure BOC is a given angle; and one of its arms BO is produced to A ; the adjacent supplementary angles, AOC , BOC , are bisected by EO , DO .

DO and EO are called respectively the *internal* and *external bisectors* of the angle BOC .

Hence Exercise 2 may be stated thus:

Prove that the internal and external bisectors of an angle are at right angles to each other.



3. Show that the angles AOE and BOD are complementary.

4. Show that the angles AOD and COD are supplementary; and also that the angles BOE and COE are supplementary.

5. If the angle BOD is 37° , how many degrees are there in AOE?

6. If two angles are supplementary, and the greater is 9 times the less, how many degrees are there in each angle?

7. If an angle is 11 times its complement, how many degrees does it contain?

PARALLEL LINES.

68. *Parallel straight lines* are such as lie in the same plane, and never meet, however far they are produced in both directions.

69. A straight line crossing several other lines is called a *transversal*; as EF.

When two straight lines are cut by a transversal, eight angles are formed, which are named as follows:

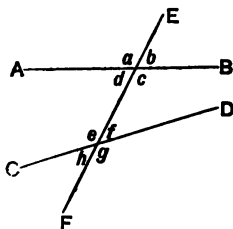
The four angles a, b, g, h , without the two lines, are called *exterior angles*.

The four angles c, d, e, f , within the two lines, are called *interior angles*.

The pair c and e , and the pair d and f , are called *alternate-interior angles*.

The pair a and g , and the pair b and h , are called *alternate-exterior angles*.

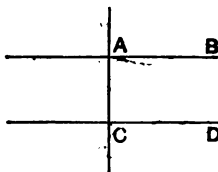
The pairs a and e , b and f , c and g , d and h , are called *corresponding angles*.*



* Called also *exterior-interior angles*.

Proposition 10. Theorem.

70. *Two straight lines in the same plane, and perpendicular to a third straight line, are parallel to each other.*



Hyp. Let AB and CD be $\perp$ to AC.

To prove AB and CD parallel.

Proof. If AB and CD are not parallel, they will meet if sufficiently produced, and we shall have two perpendiculars from the same point to the same straight line, which is impossible. (58)

Therefore they cannot meet.

$\therefore$ AB and CD are parallel. (68)

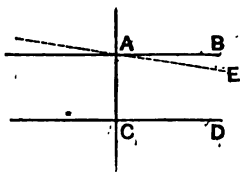
Q.E.D.

EXERCISES.

1. Find the number of degrees in each of two angles if they are complementary, and the greater is three times the less.
2. Find the number of degrees in each of two angles if they are supplementary, and the greater exceeds the less by 40° .
3. Find the number of degrees in each of two angles if they are supplementary, and the less is one-fifth the greater.

Proposition 11. Theorem.

71. *If a straight line is perpendicular to one of two parallels, it is also perpendicular to the other.*



Hyp. Let AB and CD be two parallel lines, and let AC be $\perp$ to CD.

To prove AC $\perp$ to AB.

Proof. Through A where AC intersects AB, draw AE $\perp$ to AC.

Because AE and CD are both $\perp$ to AC,

$\therefore$ AE is parallel to CD. (70)

But AB is parallel to CD. (Hyp.)

$\therefore$ AE coincides with AB.

Through a given pt. only one line can be drawn $\parallel$ to a given line (Ax. 12).

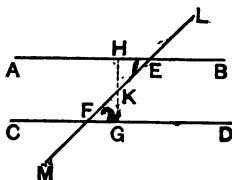
$\therefore$ AB is $\perp$ to AC, and $\therefore$ AC is $\perp$ to AB. Q.E.D.

EXERCISES.

1. Find the value of an angle if it is four times its complement.
2. Find the value of an angle if it is three times its supplement.
3. Find the value of an angle if it is one-eighth of its complement.
4. Find the number of degrees in each of two angles if they are supplementary, and the greater is four times the less.

Proposition 12. Theorem.

72. *If a straight line cut two parallel straight lines, the alternate-interior angles are equal.*



Hyp. Let the straight line LM cut the $\parallel$ straight lines AB, CD, at the points E, F.

To prove $\angle AEF = \angle EFD$.

Proof. Through K, the middle point of EF, draw $HG \perp$ to AB.

HG is also $\perp$ to CD. (71)

Turn the figure KFGD, in its own plane, about K as a pivot until KG coincides with KH.

Then since $\angle FKG = \angle EKH$, being vertical $\angle$ s, (55)

and $KF = KE$, (Cons.)

$\therefore$ KF will coincide with KE,

and point F will coincide with point E.

$\therefore$ GF $\perp$ to KG will coincide with HE $\perp$ to KH. (58)

$\therefore$ $\angle$ s KFG and KEH coincide;

$\therefore$ $\angle KFG = \angle KEH$.

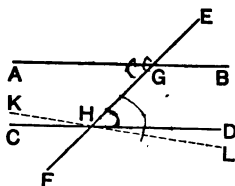
Q. E. D.

73. COR. 1. *The alternate-interior angles BEF, EFC are also equal.* (50)

74. COR. 2. *The alternate-exterior angles BEL, CFM are also equal.* (55) and (Ax. 1)

Proposition 13. Theorem.

75. *Conversely, if a straight line cut two other straight lines, so as to make the alternate-interior angles equal, the two straight lines are parallel.*



Hyp. Let the straight line EF cut the two straight lines AB, CD at G and H, so that $\angle AGH = \angle GHD$.

To prove AB parallel to CD.

Proof. Through H draw KL $\parallel$ to AB.

Then, since AB and KL are $\parallel$,

$$\angle AGH = \angle GHL. \quad (72)$$

$$\text{But} \quad \angle AGH = \angle GHD. \quad (\text{Hyp.})$$

$$\therefore \angle GHL = \angle GHD. \quad (\text{Ax. 1})$$

$\therefore$ KL and CD coincide.

$\therefore$ CD is $\parallel$ to AB.

Q.E.D.

76. COR. *If the alternate-exterior angles are equal, the two lines are parallel.*

If the sum of the two interior angles on the same side is less than two right angles, the lines will meet.

EXERCISE.

If a line is drawn through the vertex of an angle perpendicular to the bisector, prove (1) that it bisects the supplementary angle, and (2) that it makes equal angles with the sides of the given angle.

Proposition 14. Theorem.

77. *If a straight line cut two parallel straight lines, (1) any exterior angle is equal to the interior opposite angle,* and (2) the sum of the two interior angles on the same side is equal to two right angles.*

Hyp. Let the st. line EF cut the two || st. lines AB, CD at G and H.

(1) *To prove* $\angle EGB = \angle GHD$.

Proof. Since $\angle AGH = \angle GHD$,
being alt.-int. $\angle s$ (72),

and $\angle AGH = \angle EGB$,
being vertical $\angle s$ (55),

$\therefore \angle EGB = \angle GHD$. (Ax. 1)

(2) *To prove* $\angle BGH + \angle GHD = 2 \text{ rt. } \angle s$.

Proof. $\angle EGB = \angle GHD$. (Just proved)

Add $\angle BGH$ to each.

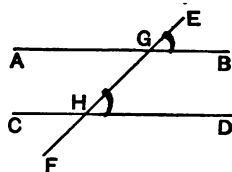
$\therefore \angle EGB + \angle BGH = \angle GHD + \angle BGH$. (Ax. 2)

But $\angle EGB + \angle BGH = 2 \text{ rt. } \angle s$,

being sup.-adj. $\angle s$ (53).

$\therefore \angle GHD + \angle BGH = 2 \text{ rt. } \angle s$. (Ax. 1)

Q.E.D.

**EXERCISES.**

1. If $\angle EGB = 42^\circ$, how many degrees are there in $\angle CHF$? in $\angle FHD$? in $\angle BGH$?

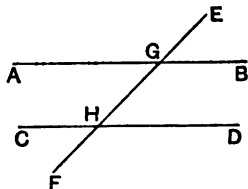
2. If two angles are supplementary, and the greater exceeds the less by 35° , how many degrees are there in each angle?

* Called corresponding, or exterior-interior, angles. (69.)

Proposition 15. Theorem.

78. *Conversely, if a straight line cut two other straight lines, (1) so as to make the exterior-interior angles equal, or (2) so as to make the two interior angles on the same side together equal to two right angles, then the two straight lines are parallel.*

(1) *Hyp.* Let the st. line EF cut the two st. lines AB, CD at G and H, so that $\angle EGB = \angle GHD$.



To prove $AB \parallel$ to CD .

Proof. Since $\angle EGB = \angle GHD$, (Hyp.)

and $\angle EGB = \angle AGH$,
being vertical $\angle$ s (55),

$\therefore \angle AGH = \angle GHD$. (Ax. 1)

But these are alternate $\angle$ s.

$\therefore AB$ is $\parallel$ to CD . (75)

(2) *Hyp.* Let $\angle BGH + \angle GHD = 2$ rt. $\angle$ s.

To prove $AB \parallel$ to CD .

Proof. Since $\angle BGH + \angle GHD = 2$ rt. $\angle$ s, (Hyp.)

and $\angle BGH + \angle AGH = 2$ rt. $\angle$ s,
being sup.-adj. $\angle$ s (52),

$\therefore \angle BGH + \angle GHD = \angle BGH + \angle AGH$. (Ax. 1)

Take away the common $\angle BGH$.

$\therefore \angle GHD = \angle AGH$. (Ax. 3)

And these being alternate $\angle$ s,

$\therefore AB$ is $\parallel$ to CD . (75)

Q. E. D.

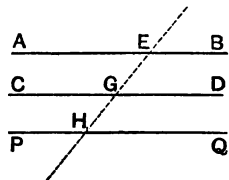
Proposition 16. Theorem.

79. *Straight lines which are parallel to the same straight line are parallel to one another.*

Hyp. Let the st. lines AB, CD be each $\parallel$ to the st. line PQ.

To prove AB $\parallel$ to CD.

Proof. Draw any st. line EGH, cutting the lines in E, G, and H.



Since AB is $\parallel$ to PQ, (Hyp.)

$$\therefore \angle AEH = \angle EHQ,$$

being alt. $\angle$ s (72).

And since CD is $\parallel$ to PQ, (Hyp.)

$$\therefore \angle EGD = \angle GHQ,$$

being ext.-int. $\angle$ s (77).

$$\therefore \angle AEH = \angle EGD, \quad (\text{Ax. 1})$$

and these are alternate $\angle$ s;

$$\therefore AB \text{ is } \parallel \text{ to } CD. \quad (75)$$

Q. E. D.

NOTE.—If PQ lies *between* AB and CD, the Proposition may be proved in a similar way, though in this case it scarcely needs proof; for it is inconceivable that two straight lines, which do not meet an intermediate straight line, should meet each other.

EXERCISES.

1. Two straight lines AB, CD, bisect each other at O. Show that the straight lines joining AC and BD are parallel.

2. If a straight line meets two parallel straight lines, and the two interior angles on the same side are bisected; show that the bisectors meet at right angles.

Proposition 17. Theorem.

80. Two angles having their sides parallel each to each, are either equal or supplementary.

Hyp. Let AB be $\parallel$ to DH, and BC be $\parallel$ to MF.

To prove $\angle ABC = \angle DEF$, and
supplementary to $\angle DEM$.

Proof. Produce BC and DH, if necessary, until they intersect in G.

'Then since the $\parallel$ s AB and DE are cut by BC,

$$\therefore \angle ABC = \angle DGC. \quad (77)$$

And since the $\parallel$ s BC and MF are cut by DH,

$$\therefore \angle \text{DGC} = \angle \text{DEF}. \quad (77)$$

$$\therefore \angle ABC = \angle DEF. \quad (\text{Ax. 1})$$

$$\angle DEF \text{ is the supplement of } \angle DEM. \quad (25)$$

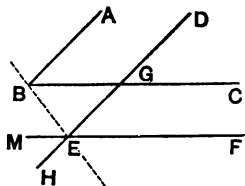
$\therefore \angle ABC$ is supplementary to $\angle DEM$. Q.E.D.

81. COR. Since $\angle DEF = \angle MEH$, being vertical $\angle$ s, (55)

$\therefore \angle ABC = \angle MEH$, and is supplementary to $\angle HEF$.

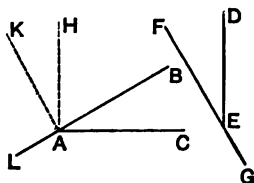
82. SCH. Two parallels are said to be in the *same* direction, or in *opposite* directions, according as they lie on the same side or on opposite sides of the straight line joining the vertices. Thus AB and ED, and also BC and EF, are in the same direction because they lie on the same side of BE. But BA and EH, and also BC and EM, are in opposite directions.

Hence, when both pairs of parallel sides are either in the same direction, or in opposite directions, the angles are *equal*. But when two of the parallel sides are in the same direction, and the other two in opposite directions, the angles are *supplementary*.



Proposition 18. Theorem.

83. *Two angles having their sides perpendicular each to each, are either equal or supplementary.*



Hyp. Let AB be $\perp$ to GF , and $AC \perp$ to DE .

To prove $\angle BAC = \angle DEF$, and supplementary to $\angle DEG$.

Proof. Draw $AH \perp$ to AC , and $AK \perp$ to AB .

Then AH is $\parallel$ to ED , and AK is $\parallel$ to EF .

Two lines $\perp$ to the same st. line are $\parallel$ (70).

Because AH and AK are respectively $\parallel$ to ED and EF , and extend in the same direction,

$$\therefore \angle HAK = \angle DEF. \quad (80)$$

But $\angle HAK$ is the complement of $\angle HAB$; (Cons.)

and $\angle BAC$ is the complement of $\angle HAB$; (Cons.)

$$\therefore \angle HAK = \angle BAC.$$

Complements of equal $\angle$ s are equal (50).

$$\therefore \angle BAC = \angle DEF. \quad (\text{Ax. 1})$$

Since $\angle DEF$ is the supplement of $\angle DEG$,

$$\therefore \angle BAC \text{ is supplementary to } \angle DEG. \quad \text{Q.E.D.}$$

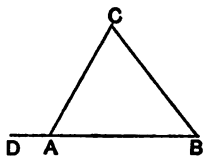
84. SCH. If both angles are acute or both obtuse, they are *equal*; but if one is acute and the other obtuse, they are *supplementary*.

TRIANGLES.

85. A *triangle* is a plane figure bounded by three straight lines.

The three straight lines which bound a triangle are called its *sides*. Thus, AB, BC, CA, are the sides of the triangle ABC.

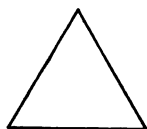
The *angles* of the triangle are the angles formed by the sides with each other; as BAC, ABC, ACB. The vertices of these angles are also called the *vertices* of the triangle.



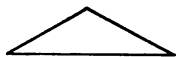
86. An *exterior angle* of a triangle is the angle formed between any side and the continuation of another side; as CAD.

The angles BAC, ABC, BCA are called *interior angles* of the triangle. When we speak of the angles of a triangle we mean the three interior angles.

87. An *equilateral* triangle is one whose three sides are equal.



88. An *isosceles* triangle is one which has two equal sides.



89. A *scalene* triangle is one which has three unequal sides.



90. A *right-angled* triangle is one which has a right angle.

The side opposite the right angle is called the *hypotenuse*, and the other two sides the *legs*.



91. An *acute-angled* triangle is one which has *three* acute angles.



It will be shown hereafter that every triangle must have at least two acute angles.

92. An *obtuse-angled* triangle is one which has an obtuse angle.

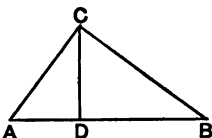


When all the angles are equal it is called an *equiangular* triangle.

93. The *base* of a triangle is the side upon which it is supposed to stand.

Either side may be taken as the base; but in an *isosceles* triangle the side which is not one of the equal sides is called the base.

94. When one side of a triangle has been taken as the base, the angle opposite is called the *vertical angle*, and its vertex is called the vertex of the triangle.



95. The *altitude* of a triangle is the perpendicular let fall from the vertex A to the base, or the base produced.

Thus, in the triangle ABC, AB is the base, ACB is the vertical angle, C is the vertex, and CD is the altitude.

When two triangles have the angles of the one equal respectively to the angles of the other, the angles which are equal are called *homologous angles*, and the sides which are opposite the equal angles are called *homologous sides*; hence in equal triangles the homologous sides are equal.

Proposition 19. Theorem.

96. *Either side of a triangle is less than the sum, and greater than the difference, of the other two sides.*

In the $\triangle ABC$,

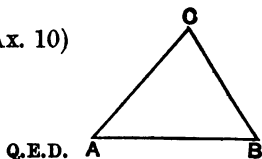
$$AB < AC + BC, \quad (\text{Ax. 10})$$

Taking AC from each,

$$AB - AC < BC,$$

or

$$BC > AB - AC.$$



Proposition 20. Theorem.

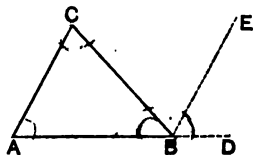
97. *The sum of the three angles of any triangle is equal to two right angles.*

Hyp. Let ABC be any $\triangle$.

To prove

$$\angle A + \angle ABC + \angle C = 2 \text{ rt. } \angle \text{s.}$$

Proof. Produce AB to D ,
and draw $BE \parallel$ to AC .



Then since BE is $\parallel$ to AC ,

$$\therefore \text{ext. } \angle EBD = \text{int. } \angle A, \quad (77)$$

$$\text{and} \quad \angle CBE = \text{alt. } \angle C. \quad (72)$$

$$\text{Add} \quad \angle ABC \text{ to each.}$$

$$\therefore \angle ABC + \angle CBE + \angle EBD = \angle ABC + \angle C + \angle A. \quad (\text{Ax. } 2)$$

$$\text{But} \quad \angle ABC + \angle CBE + \angle EBD = 2 \text{ rt. } \angle \text{s.}$$

The sum of all the $\angle$ s on the same side of a st. line at a point = 2 rt. $\angle$ s (58).

$$\therefore \angle ABC + \angle C + \angle A = 2 \text{ rt. } \angle \text{s.} \quad (\text{Ax. } 1)$$

Q.E.D.

98. COR. 1. Since $\angle EBD = \angle A$, and $\angle CBE = \angle C$,

$$\therefore \angle CBD = \angle A + \angle C.$$

Hence, the exterior angle of a triangle is equal to the sum of the two opposite interior angles.

99. COR. 2. *If two angles of a triangle are given, or merely their sum, the third angle can be found by subtracting this sum from two right angles.*

100. COR. 3. *If two triangles have two angles of the one equal to two angles of the other, the third angles are equal.*

101. COR. 4. *A triangle can have but one right angle, or but one obtuse angle.*

102. COR. 5. *In any right-angled triangle the two acute angles are complementary.*

103. COR. 6. *Each angle of an equiangular triangle is two-thirds of a right angle.*

Proposition 21. Theorem.

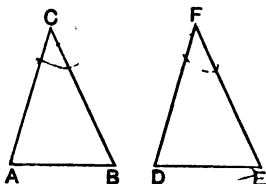
104. *Two triangles are equal when two sides and the included angle of the one are equal respectively to two sides and the included angle of the other.*

Hyp. Let $\triangle ABC$, $\triangle DEF$ be two $\triangle$ s, having $AB = DE$, $AC = DF$.

$$\angle A = \angle D.$$

To prove $\triangle ABC = \triangle DEF$.

Proof. Apply the $\triangle ABC$ to the $\triangle DEF$ (29) so that the point A shall fall on D , and the side AB on DE .



Then because $AB = DE$, (Hyp.)

$\therefore$ pt. B must fall on E .

Because $\angle A = \angle D$, (Hyp.)

$\therefore$ AC must lie along DF .

And because $AC = DF$, (Hyp.)

$\therefore$ pt. C must fall on F .

Now, since B falls on E , and C on F ,

$\therefore$ BC must coincide with EF . (Ax. 11)

Hence, since BC coincides with EF ,

$$\therefore BC = EF, \angle B = \angle E, \angle C = \angle F.$$

Therefore the two $\triangle$ s coincide in all their parts, and hence are equal (29). Q.E.D.

NOTE.—When all the parts of one triangle are respectively equal to all the parts of another triangle, the triangles are said to be “equal in all respects.” Such triangles are said to be *identically equal*, or *congruent*.

This proposition is proved by “the method of superposition,” i.e., it is shown that one of the triangles could be placed on the other so as to cover it exactly without overlapping; and then, since all their parts would *coincide*, the two triangles are equal by definition (29).

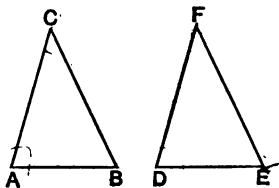
Proposition 22. Theorem.

105. *Two triangles are equal when a side and the two adjacent angles of the one are equal respectively to a side and the two adjacent angles of the other.*

Hyp. Let $\triangle ABC$, $\triangle DEF$ be two $\triangle$ s having $\angle A = \angle D$, $\angle B = \angle E$, $AB = DE$.

To prove $\triangle ABC = \triangle DEF$.

Proof. Apply the $\triangle ABC$ to the $\triangle DEF$ so that the point A shall fall on D , and the side AB on DE .



Then, because $AB = DE$, (Hyp.)

$\therefore$ pt. B must fall on E .

Because $\angle A = \angle D$, (Hyp.)

$\therefore$ AC must lie along DF .

Because $\angle B = \angle E$, (Hyp.)

$\therefore$ BC must lie along EF .

Because AC and BC fall upon DF and EF , respectively, $\therefore$ the point C , falling upon both lines DF and EF , must fall at their point of intersection F .

$\therefore$ the two $\triangle$ s coincide in all their parts, and are equal.

Q.E.D.

106. COR. 1. *Two right-angled triangles are equal when the hypotenuse and an acute angle of the one are equal respectively to the hypotenuse and an acute angle of the other.*

107. COR. 2. *Two right-angled triangles are equal when a side and an acute angle of the one are equal respectively to a side and homologous acute angle of the other.*

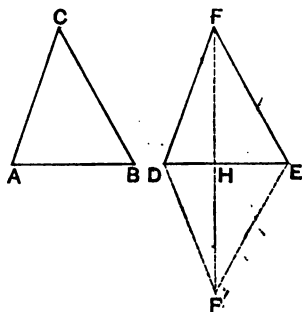
Proposition 23. Theorem.

108. *Two triangles are equal if the three sides of the one are equal respectively to the three sides of the other.*

Hyp. Let $\triangle ABC$, $\triangle DEF$ be two $\triangle$ s having $AB = DE$, $AC = DF$, $BC = EF$.

To prove $\triangle ABC = \triangle DEF$.

Proof. Apply the $\triangle ABC$ to the $\triangle DEF$ so that the side AB shall coincide with its equal DE , and the vertex C fall at F' on the opposite side of DE from F .



Join FF' which cuts DE at H .

Because $DF = DF'$, and $EF = EF'$, (Hyp.)

$\therefore$ points D and E are equally distant from F and F' .

$\therefore DE$ is $\perp$ to FF' at its middle pt. H . (67)

Now revolve the $\triangle DEF$ about DE as an axis till it comes into the plane of the $\triangle DEF'$.

Because $\angle DHF = \angle DHF' = \text{a rt. } \angle$ (67)

and $HF = HF'$,

$\therefore$ pt. F will fall on F' ,

$\therefore$ the two $\triangle$ s coincide in all their parts, and are equal.

Q. E. D.

109. SCH. When two triangles are equal, the equal angles lie opposite the equal sides; and conversely, the equal sides lie opposite the equal angles. Thus, the angles A and D are equal or homologous angles (95). Also the angles C and F are homologous; likewise the angles B and E .

Proposition 24. Theorem.

110. *Two right-angled triangles are equal when the hypotenuse and a side of the one are equal respectively to the hypotenuse and a side of the other.*



Hyp. Let ABC , DEF be two rt. Δ s having

hyp. $AC = \text{hyp. } DF$, and $BC = EF$.

To prove $\Delta ABC = \Delta DEF$.

Proof. Apply the ΔABC to the ΔDEF so that BC shall coincide with its equal EF .

Then, since $\angle B = \angle E = \text{a rt. } \angle$,

$\therefore BA$ will lie along ED , and pt. A will fall somewhere on ED .

But the equal oblique lines CA and FD cut off on ED equal distances from the foot of the $\perp EF$. (61)

$\therefore$ pt. A will fall at D .

$\therefore$ the two Δ s coincide in all their parts, and are equal.

Q.E.D.

EXERCISES.

1. If BC , the base of an isosceles triangle, ABC , is produced to any point D , show that AD is greater than either of the equal sides.

2. Prove that the sum of the distances of any point from the three vertices of a triangle is greater than half its perimeter.

3. If one of the acute angles of a right triangle is $40^\circ 14' 48''$, what is the value of the other acute angle?

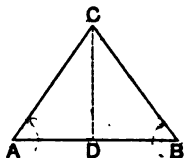
Proposition 25. Theorem.

111. *In an isosceles triangle the angles opposite the equal sides are equal.*

Hyp. Let ABC be an isosceles $\triangle$ having
 $AC = BC$.

To prove $\angle A = \angle B$.

Proof. Draw the line CD from the vertex C to the middle pt. D of the base.



Then, in the $\triangle$ s ADC and BDC ,

because $AC = BC$, (Hyp.)

$AD = DB$, (Cons.)

$CD = CD$, (being common to both $\triangle$ s)

$\therefore \triangle ADC = \triangle BDC$.

Two $\triangle$ s are equal if the three sides are equal each to each (108).

$\therefore \angle A = \angle B$,

being opposite equal sides (109).

Q.E.D.

112. **COR. 1.** Since $\triangle ADC = \triangle BDC$, we have $\angle ACD = \angle BCD$, and $\angle ADC = \angle BDC$. Therefore, *the straight line which joins the vertex to the middle point of the base of an isosceles triangle, is at right angles to the base, and bisects the vertical angle.*

Hence, also, *the bisector of the vertical angle of an isosceles triangle bisects the base at right angles.*

113. **COR. 2.** *The perpendicular from the vertex to the base of an isosceles triangle bisects the base and the angle at the vertex.*

An equilateral triangle is also equiangular.

Proposition 26. Theorem.

114. *Conversely, if two angles of a triangle are equal, the sides opposite the equal angles are equal, and the triangle is isosceles.*

Hyp. Let ABC be a $\triangle$ having

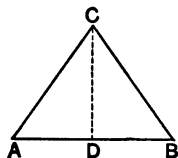
$$\angle A = \angle B.$$

To prove

$$AC = BC.$$

Proof. Draw CD, bisecting the $\angle ACB$.

Then, in $\triangle$ s ACD, BCD,



$$\angle A = \angle B. \quad (\text{Hyp.})$$

$$\angle ACD = \angle BCD. \quad (\text{Cons.})$$

$$\therefore \angle ADC = \angle BDC. \quad (100)$$

Also, CD is common.

$$\therefore \triangle ACD = \triangle BCD,$$

having a side and the two adjacent angles equal, each to each (105).

$$\therefore AC = BC,$$

being homologous sides of equal $\triangle$ s (109). Q.E.D.

115. COR. *An equiangular triangle is also equilateral.*

EXERCISES.

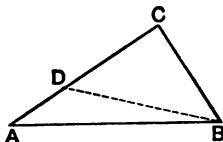
1. ABC is an isosceles triangle, with $AB = AC$. The bisectors of the angles B and C meet at O. Prove that $CO = BO$.

2. ABC is a triangle; BA is produced to D so that $AD = AC$, and DC is joined. Prove that $\angle BCD > \angle BDC$.

3. The angle C is twice as large as either of the angles A and B: how many degrees are there in each angle?

Proposition 27. Theorem.

116. *If one side of a triangle be greater than another, the angle opposite the greater side is greater than the angle opposite the less.*



Hyp. Let ABC be a $\triangle$ having the side
 $AC > BC$.

To prove $\angle ABC > \angle A$.

Proof. From CA cut off $CD = CB$.

Join BD .

Then, since $CD = CB$, (Cons.)

$$\therefore \angle CDB = \angle CBD. \quad (111)$$

But ext. $\angle CDB >$ opposite int. $\angle DAB$ of $\triangle ADB$. (98)

$$\therefore \text{also } \angle CBD > \angle DAB.$$

$$\text{But } \angle CBA > \angle CBD. \quad (\text{Ax. } 8)$$

$$\text{Much more } \therefore \angle CBA > \angle DAB. \quad \text{Q.E.D.}$$

EXERCISES.

1. Prove Prop. 27 by producing CB to E , making $CE = CA$.

2. ABC is a triangle in which OB , OC bisect the angles B , C , respectively; show that, if AB is greater than AC , then OB is greater than OC .

Proposition 28. Theorem.

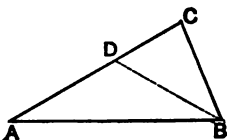
117. *Conversely, if one angle of a triangle be greater than another, the side opposite the greater angle is greater than the side opposite the less.*

Hyp. Let ABC be a $\triangle$ having
 $\angle ABC > \angle BAC$.

To prove $AC > BC$.

Proof. Draw BD so that

$$\angle ABD = \angle BAD.$$



Then $AD = BD$,
being opposite equal $\angle$ s (114).

But in $\triangle BCD$, $BD + DC > BC$.

Either side of a $\triangle$ is $<$ the sum of the other two (96).

And, since $AD = BD$,

$$\therefore BD + DC = AC.$$

$$\therefore AC > BC.$$

Q.E.D.

NOTE.—These two propositions may be stated briefly as follows: *In every triangle, the greater side is opposite the greater angle, and conversely, the greater angle is opposite the greater side.*

118. COR. *The hypotenuse is the greatest side of a right-angled triangle.*

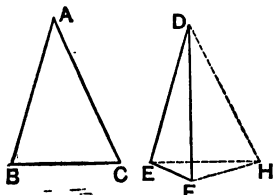
EXERCISES.

1. In a $\triangle ABC$, if AC is not greater than AB , show that any straight line drawn through the vertex A and terminated by the base BC , is less than AB .

2. Any two sides of a triangle are together greater than twice the straight line drawn from the vertex to the middle point of the third side.

Proposition 29. Theorem.

119. *If two triangles have two sides of the one equal respectively to two sides of the other, but the included angle in the first triangle greater than the included angle in the second, then the third side of the first triangle is greater than the third side of the second.*



Hyp. Let ABC , DEF be two Δ s, having
 $AB = DE$, $AC = DF$,

but $\angle BAC > \angle EDF$.

To prove $BC > EF$.

Proof. Apply the ΔABC to the ΔDEF so that AB shall coincide with DE , and the pt. C fall at H .

Join FH .

Then, since $DH = DF$, (Hyp.)

$$\therefore \angle DHF = \angle DFH. \quad (111)$$

$$\text{But } \angle DHF > \angle EHF.$$

The whole is greater than any of its parts (Ax. 8).

$$\therefore \text{also } \angle DFH > \angle EHF.$$

$$\text{But } \angle EFH > \angle DFH. \quad (\text{Ax. 8})$$

$$\text{Much more } \therefore \angle EFH > \angle EHF.$$

$$\therefore EH > EF.$$

The greater side is opposite the greater angle (117).

$$\text{But } EH = BC. \therefore BC > EF. \quad \text{Q. E. D.}$$

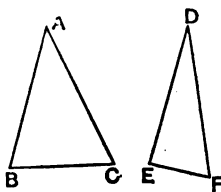
Proposition 30. Theorem.

120. *Conversely, if two triangles have two sides of the one equal respectively to two sides of the other, but the third side of the first triangle greater than the third side of the second, then the included angle of the first triangle is greater than the included angle of the second.*

Hyp. Let ABC , DEF be two Δ s
 having $AB = DE$, $AC = DF$,
 but $BC > EF$.

To prove $\angle A > \angle D$.

Proof. If $\angle A$ is not $> \angle D$,



then either (1) $\angle A = \angle D$,

or (2) $\angle A < \angle D$.

(1) If $\angle A = \angle D$,

then $BC = EF$,

having two sides and the included $\angle$ equal, each to each (104).

But this is contrary to the hypothesis.

(2) If $\angle A < \angle D$,

then $BC < EF$. (119)

But this is contrary to the hypothesis.

Hence, since $\angle A$ can neither be $=$ nor $< \angle D$,

$\therefore \angle A > \angle D$.

Q. E. D.

NOTE.—Prop. 30 is here proved by an *indirect* method. This method is sometimes called the *reductio ad absurdum*. It consists in assuming that the conclusion to be proved is not true, and showing that this assumption leads to an absurdity, or to a result inconsistent with the hypothesis.

QUADRILATERALS.

121. A *quadrilateral* is a plane figure bounded by four straight lines, which are called its *sides*.

The straight lines which join opposite angles of a quadrilateral are called the *diagonals*.

122. A *trapezium* is a quadrilateral which has no two of its sides parallel.

123. A *trapezoid* is a quadrilateral which has two of its sides parallel.

The parallel sides of a trapezoid are called the *bases*, and the perpendicular distance between them is called the *altitude*. The line joining the middle points of the non-parallel sides is called the *middle parallel** of the trapezoid.

124. A *parallelogram* is a quadrilateral which has its opposite sides parallel.

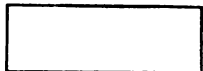
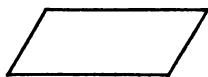
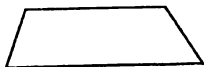
The *bases* of a parallelogram are the side on which it stands and the opposite side. The perpendicular distance between the bases is called the *altitude*.

125. A *rectangle* is a parallelogram whose angles are right angles.†

126. A *square* is a rectangle whose sides are all equal.‡

127. A *rhomboid* is a parallelogram whose angles are oblique and whose adjacent sides are unequal.

128. A *rhombus*, or *lozenge*, is a parallelogram whose sides are all equal.§



* Called also the *median*.

† Called also a *right-angled parallelogram*.

‡ Called also an *equilateral rectangle*.

§ Called also *equilateral rhomboid*.

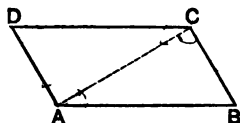
Proposition 31. Theorem.

129. *In every parallelogram, the opposite sides are equal, and the opposite angles are equal.*

Hyp. Let ABCD be a $\square$.

To prove $DC = AB$, $AD = BC$,
 $\angle A = \angle C$, $\angle D = \angle B$.

Proof. Draw the diagonal AC.



Because DC is $\parallel$ to AB , (Hyp.)

and AD is $\parallel$ to BC , (Hyp.)

$$\therefore \angle DCA = \angle BAC,$$

and $\angle ACB = \angle CAD$,
being alt.-int. $\angle$ s (72).

Hence the whole $\angle DCB = \angle BAD$. (Ax. 9)

Now in the $\triangle$ s ACD , ACB , because

$$\left\{ \begin{array}{l} \angle DCA = \angle BAC, \\ \angle CAD = \angle ACB, \end{array} \right\} \quad (\text{just proved})$$

and AC is common,

$$\therefore \triangle ACD = \triangle ACB,$$

having a side and the two adjacent $\angle$ s equal, each to each (105).

$$\therefore DC = AB, AD = BC,$$

being homologous sides of equal $\triangle$ s (109),

and $\angle D = \angle B$. Q.E.D.

130. COR. 1. *A diagonal of a parallelogram divides it into two equal triangles.*

131. COR. 2. *Two parallels included between two other parallels are equal.*

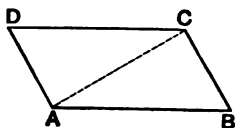
Proposition 32. Theorem.

132. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

Hyp. Let ABCD be a quadrilateral having

$$AB = CD, AD = BC.$$

To prove ABCD is a $\square$.



Proof. Draw the diagonal AC.

In the $\triangle$ s ACD, ACB, because

$$AD = BC, \quad (\text{Hyp.})$$

$$AB = CD, \quad (\text{Hyp.})$$

and

AC is common,

$$\therefore \triangle ACD = \triangle ACB. \quad (108)$$

$$\therefore \angle ACD = \angle CAB,$$

and

$$\angle CAD = \angle ACB,$$

being homologous $\angle$ s of equal $\triangle$ s (109).

$$\therefore AB \text{ is } \parallel \text{ to } CD,$$

and

$$AD \text{ is } \parallel \text{ to } BC,$$

the alt.-int. $\angle$ s being equal (75).

$$\therefore ABCD \text{ is a } \square \text{ by definition.} \quad (124)$$

Q.E.D.

EXERCISES.

1. If one angle of a parallelogram is a right angle, prove that all its angles are right angles.

2. Prove that two parallels are everywhere equally distant.

3. If, in the figure of Prop. 32, BE be drawn parallel to AC and meeting DA produced to E, prove that the parallelogram EBCA will be equal to the parallelogram ABCD.

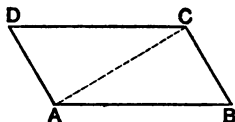
Proposition 33. Theorem.

133. *If two opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*

Hyp. Let ABCD be a quadrilateral, having $AB =$ and $\parallel$ to DC.

To prove ABCD is a $\square$.

Proof. Draw the diagonal AC.



In the $\triangle$ s ACD, ACB, because

$$AB = CD, \quad (\text{Hyp.})$$

AC is common,

$$\angle ACD = \angle CAB,$$

being alt.-int. $\angle$ s (72),

$$\therefore \triangle ACD = \triangle ACB,$$

having two sides and the included $\angle$ equal, each to each (104).

$$\therefore AD = BC,$$

being homologous sides of equal $\triangle$ s (109).

$$\therefore \text{the figure ABCD is a } \square,$$

having its opposite sides equal (132).

Q.E.D.

EXERCISES.

1. Prove that the straight lines which bisect two adjacent angles of a parallelogram cut each other at right angles.

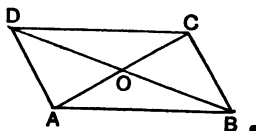
2. AB, CD, EF are three equal and parallel straight lines; prove that the triangle ACE is equal to the triangle BDF.

Proposition 34. Theorem.**134.** *The diagonals of a parallelogram bisect each other.*

Hyp. Let ABCD be a $\square$, whose diagonals intersect at O.

To prove $AO = OC$, $DO = OB$.

Proof. In the $\triangle$ s AOB, COD,



$$AB = DC,$$

being opp. sides of the $\square$ (129),

$$\angle ABO = \angle CDO,$$

and

$$\angle BAO = \angle DCO,$$

being alt.-int. $\angle$ s (72).

$$\therefore \triangle AOB = \triangle COD,$$

having a side and the two adj. $\angle$ s equal, each to each (105).

$$\therefore AO = OC, \text{ and } DO = OB. \quad (109)$$

Q.E.D.

EXERCISES.

1. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.

2. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

3. If the diagonals of a parallelogram are equal, the figure is a rectangle; if they *also* intersect at right angles, it is a square.

4. The straight lines joining the middle points of the opposite sides of any quadrilateral bisect each other.

5. The diagonals of a rhombus bisect each other at right angles.

6. The diagonals of a rectangle are equal.

Proposition 35. Theorem.

135. *Two parallelograms are equal when two adjacent sides and the included angle of the one are equal respectively to two adjacent sides and the included angle of the other.*

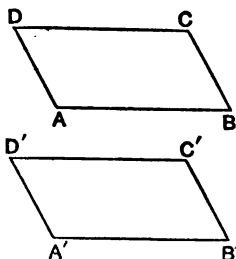
Hyp. Let $ABCD$, $A'B'C'D'$ be two
 $\square$ s, having $AB = A'B'$,

$$AD = A'D',$$

$$\angle A = \angle A'.$$

To prove $\square ABCD = \square A'B'C'D'$.

Proof. Apply $\square ABCD$ to $\square A'B'C'D'$, so that $\angle A$ shall coincide with $\angle A'$, and the side AB with the equal side $A'B'$.



Because $\angle A = \angle A'$, and $AD = A'D'$, (Hyp.)

$\therefore AD$ will fall on $A'D'$, and pt. D on pt. D' .

Because DC is $\parallel$ to AB , and $D'C'$ is $\parallel$ to $A'D'$, (124)

$\therefore DC$ will fall on $D'C'$, and pt. C somewhere on $D'C'$.

Through a given pt. only one st. l. can be drawn $\parallel$ to a given st. l. (Ax. 12).

Also, because BC is $\parallel$ to AD , and $B'C'$ is $\parallel$ to $A'D'$, (124)

$\therefore BC$ will fall on $B'C'$, and pt. C somewhere on $B'C'$. (Ax. 12)

$\therefore$ since pt. C falls on $D'C'$ and $B'C'$, it must fall at their pt. of intersection C' .

$\therefore$ the two $\square$ s coincide throughout, and are equal. Q. E. D.

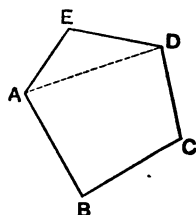
136. COR. *Two rectangles are equal when they have two adjacent sides equal, each to each.*

POLYGONS.

137. A *polygon* is a plane figure bounded by straight lines; as ABCDE.

The straight lines are called the *sides* of the polygon; and their sum is called the *perimeter* of the polygon.

The *angles* of the polygon are the angles formed by the adjacent sides with each other; and the vertices of these angles are also called the *vertices* of the polygon.



138. The angles of the polygon measured on the side of the enclosed surface are called *interior angles*.

An *exterior angle* of a polygon is an angle between any side and the continuation of an adjacent side.

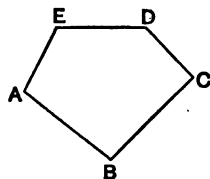
A *diagonal* is a line joining any two vertices that are not consecutive; as AD.

139. Polygons are named from the number of their sides, as follows:

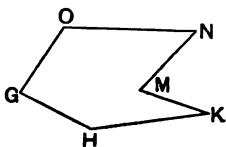
A polygon of three sides is a *triangle*; one of four sides, a *quadrilateral*; one of five sides, a *pentagon*; one of six sides, a *hexagon*; one of seven sides, a *heptagon*; one of eight sides, an *octagon*; one of nine sides, a *nonagon*; one of ten sides, a *decagon*; one of twelve sides, a *dodecagon*; one of fifteen sides, a *quindecagon*.

140. An *equilateral* polygon is one which has all its sides equal. An *equiangular* polygon is one which has all its angles equal. A *regular* polygon is one which is both equilateral and equiangular.

141. A *convex* polygon is one each of whose interior angles is less than a straight angle; as ABCDE.



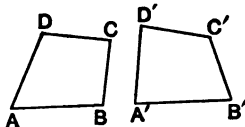
142. A *concave* polygon is one in which at least one of the interior angles is reflex (26), as GHKMNO, in which the interior angle KMN is reflex.*



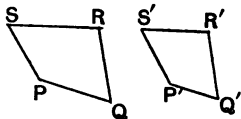
The polygons considered in this work will be understood to be convex, unless otherwise stated.

It is evident that a polygon has as many angles as sides.

143. Two polygons are *mutually equilateral* when the sides of the one are equal respectively to the sides of the other, taken in the same order; as the polygons ABCD, A'B'C'D', in which $AB = A'B'$, $BC = B'C'$, etc.



144. Two polygons are *mutually equiangular* when the angles of the one are equal respectively to the angles of the other, taken in the same order; as the polygons PQRS, P'Q'R'S', in which $\angle P = \angle P'$, $\angle Q = \angle Q'$, etc.



145. In polygons which are mutually equilateral or mutually equiangular, any two corresponding sides or angles are called *homologous* (95).

Except in the case of triangles, two polygons may be mutually equilateral without being mutually equiangular, and mutually equiangular without being mutually equilateral.

146. A polygon may be divided into triangles by drawing diagonals from one of its vertices; and the number of triangles into which any polygon can thus be divided is evidently equal to the number of its sides, less two.

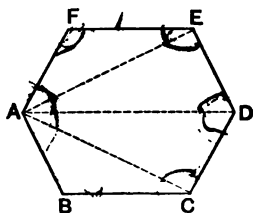
When two polygons can be divided by diagonals into the same number of triangles, equal each to each, and similarly placed, the polygons are *equal*; for they can be applied one to the other, and the corresponding triangles will evidently coincide, and therefore the polygons will coincide throughout.

When two polygons are both mutually equilateral and mutually equiangular, they are *equal*, for they can be applied one to the other so as to coincide.

* Called also a *re-entrant* angle.

Proposition 36. Theorem.

147. *The sum of the interior angles of a polygon is equal to two right angles taken as many times less two as the polygon has sides.*



Hyp. Let ABCDEF be a polygon having n sides.

To prove $\angle A + \angle B + \angle C + \text{etc.} = 2 \text{ rt. } \angle s (n - 2)$.

Proof. From any vertex A, draw the diagonals AC, AD, AE. The polygon will be divided into as many triangles less two as it has sides (146). Thus, there are $(n - 2)$ triangles, whose angles make the interior angles of the polygon.

Now, the sum of the $\angle s$ of a $\triangle = 2 \text{ rt. } \angle s$. (97)

$\therefore$ the sum of the $\angle s$ in the polygon $= 2 \text{ rt. } \angle s (n - 2)$.

Q. E. D.

148. COR. 1. $2 \text{ rt. } \angle s (n - 2) = 2n \text{ rt. } \angle s - 4 \text{ rt. } \angle s$.

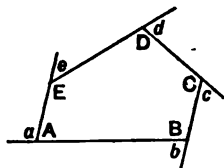
Therefore, the sum of the angles of a polygon is also equal to twice as many right angles as the figure has sides, less four right angles.

149. COR. 2. The sum of the angles of a quadrilateral is equal to two right angles taken $(4 - 2)$ times, i.e., four right angles. The sum of the angles of a pentagon is equal to two right angles taken $(5 - 2)$ times, i.e., six right angles, etc.

150. COR. 3. Each angle of an equiangular polygon of n sides is $\frac{2(n - 2)}{n}$ right angles.

Proposition 37. Theorem.

151. *The sum of the exterior angles of a polygon, made by producing each of its sides in the same direction, is equal to four right angles.*



Hyp. Let ABCDE be a polygon having its n sides produced in the same direction.

To prove $\angle a + \angle b + \angle c + \text{etc.} = 4 \text{ rt. } \angle \text{s.}$

Proof. Any int. $\angle A$ + its adj.-ext. $\angle a = 2 \text{ rt. } \angle \text{s.}$ (52)

$\therefore$ all the int. $\angle \text{s}$ + all the ext. $\angle \text{s} = 2n \text{ rt. } \angle \text{s.}$

But all the int. $\angle \text{s} = 2n \text{ rt. } \angle \text{s} - 4 \text{ rt. } \angle \text{s.}$ (148)

$\therefore$ all the ext. $\angle \text{s} = 4 \text{ rt. } \angle \text{s.}$ (Ax. 3)

Q.E.D.

EXERCISES.

1. Express in terms of a right angle, and also in degrees, the magnitude of each interior angle of (1) a regular hexagon, (2) a regular octagon, and (3) a regular decagon.

2. If one side of a regular hexagon is produced, show that the exterior angle is equal to the angle of an equilateral triangle.

3. The exterior angle of a regular polygon is one-fifth of a right angle: find the number of sides in the polygon.

4. The interior angle of a regular polygon is five-thirds of a right angle: find the number of sides in the polygon.

5. The side AB of the triangle ABC is produced to D, so that BD is equal to BC: prove that the angle ABC is double the angle ADC.

6. How many sides has a polygon, the sum of whose interior angles is four times that of its exterior angles?

MISCELLANEOUS THEOREMS.

Proposition 38. Theorem.

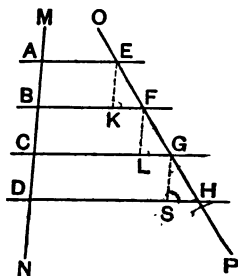
152. *If three or more parallels intercept equal lengths on any transversal, they intercept equal lengths on every transversal.*

Hyp. Let the $\parallel$ s AE, BF, CG, DH, cut the two transversals MN and OP at the pts. A, B, C, D, and E, F, G, H, so that

$$AB = BC = CD = \text{etc.}$$

To prove $EF = FG = GH = \text{etc.}$

Proof. From E, F, G, draw EK, FL, GS, all $\parallel$ to MN.



Then $EK = AB$, $FL = BC$, $GS = CD$,
being opp. sides of a $\square$ (129).

$$\therefore EK = FL = GS = \text{etc.} \quad (\text{Ax. 1})$$

Also, $\angle FEK = \angle GFL = \angle HGS$,

and $\angle EFK = \angle FGL = \angle GHS$,
being ext.-int. $\angle$ s of $\parallel$ lines (77).

$\therefore \triangle EFK = \triangle FGL = \triangle GHS$,
having a side and the two adj. $\angle$ s equal, each to each (100, 105).

$\therefore EF = FG = GH$,
being homologous sides of equal $\triangle$ s (109). Q.E.D.

153. COR. Since $FK = GL = HS$,

$$\therefore BF - AE = FK,$$

and $CG - BF = GL = FK.$

Therefore, the intercepted part of each parallel will differ in length from the next intercept by the same amount.

Proposition 39. Theorem.

154. *The straight line drawn through the middle point of a side of a triangle parallel to the base, bisects the remaining side, and is equal to half the base.*

Hyp. Let ABC be a $\triangle$, D the middle point of AB , and DE a line $\parallel$ to BC , meeting AC at E .

To prove (1) $AE = EC$;

and (2) $DE = \frac{1}{2}BC$.

Proof. (1) Through A draw a line $\parallel$ to BC .

Then, because $AD = DB$, (Hyp.)
 $\therefore AE = EC$.

If $\parallel$ s intercept equal lengths on one transversal, they intercept equal lengths on every transversal (152).

(2) The intercepted part BC exceeds DE as much as DE exceeds the intercepted part at A (153). But this latter intercept $= 0$.

$$\therefore BC - DE = DE.$$

$$\therefore DE = \frac{1}{2}BC.$$

Q. E. D.

155. COR. 1. *The line joining the middle points of two sides of a triangle is parallel to the third side, and equal to half of it.*

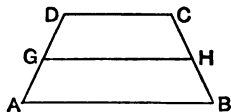
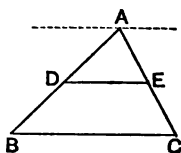
156. COR. 2. Let $ABCD$ be a trapezoid, $AG = GD$, and $GH \parallel$ to AB .

Then $BH = HC$, for, since AB , GH , and DC are parallels intercepting equal lengths AG and GD on the transversal AD , they intercept equal lengths on every transversal. (152)

Also, $AB - GH = GH - DC$. (153)

$$\therefore GH = \frac{1}{2}(AB + DC).$$

Therefore, *the line drawn through the middle point of one of the non-parallel sides of a trapezoid parallel to the bases, bisects the opposite side, and is equal to half their sum.*



THE LOCUS OF A POINT.

157. The *locus of a point* is the line, or system of lines, which contains all the points, and only those points, that satisfy a given condition.

A locus is sometimes defined as the path traced out by a point which moves in accordance with a given law.

Thus, the locus of a point which is always at a given distance from a given straight line, is a pair of straight lines.

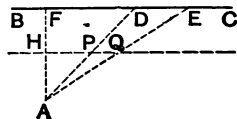
158. In order to infer that a certain line, or system of lines, is the locus of a point under a given condition, it is necessary to prove, (1) *that any point which fulfils the given condition is on the supposed locus*; and (2) *that every point on the supposed locus satisfies the given condition*.

159. *The locus of a point which is always equidistant from two given points, is the perpendicular bisector of the line joining them (67).*

EXERCISE.

Find the locus of the middle point of a straight line drawn from a given point to meet a given straight line of unlimited length.

Let A be the given pt., and BC the given st. line.



(1) Let AD be any straight line drawn from A to meet BC, and let P be its mid. pt.

Draw $AF \perp$ to BC, and bisect AF at H.

Join HP, and produce it indefinitely.

Then HP is $\parallel$ to FD . (155)

$\therefore P$ is on the st. line which passes through the fixed pt. H, and is $\parallel$ to BC.

(2) Every pt. in HP, or HP produced, satisfies the required condition.

For in this st. line take any pt. Q. Join AQ, and produce it to meet BC in E.

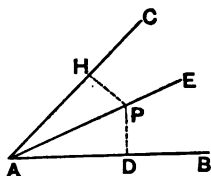
Then Q is the mid. pt. of AE. (154)

Hence the required locus is the st. line $\parallel$ to BC and passing through the mid. pt. of the $\perp$ from A to BC.

Proposition 40. Theorem.

160. (1) *Every point in the bisector of an angle is equally distant from the sides of the angle; and (2), conversely, every point within an angle, and equally distant from its sides, is in the bisector of the angle.*

(1) *Hyp.* Let AE be the bisector of the $\angle BAC$; P any point in AE ; and $PD, PH \perp$ s to AB, AC .



To prove $PD = PH.$ (59)

Proof. Because in the rt. Δ s $APD, APH,$

$$\angle DAP = \angle HAP, \quad (\text{Hyp.})$$

and

AP is common,

$$\therefore \Delta APD = \Delta APH,$$

having the hypotenuse and an acute $\angle$ equal in each (106).

$$\therefore PD = PH.$$

being homologous sides of equal Δ s (109).

(2) *Hyp.* Let $PD \perp$ to $AB = PH \perp$ to AC .

To prove that P is on the bisector of $\angle BAC$.

Proof. Join PA .

Then $\Delta APD = \Delta APH,$

having the hypotenuse and a side equal in each (110).

$$\therefore \angle PAD = \angle PAH,$$

being homologous $\angle$ s of equal Δ s (109).

$\therefore P$ is on the bisector of the angle $BAC.$ Q.E.D.

161. COR. *Every point within the angle, but not on the bisector, is unequally distant from the two sides.*

162. SCH. *The bisector of an angle is the locus of all the points situated within the angle, which are equally distant from its sides.*

THE CONCURRENCE OF STRAIGHT LINES IN A TRIANGLE.

163. Three or more straight lines are said to be *concurrent* when they meet in a point.

164. Three or more points are said to be *collinear* when they lie upon one straight line.

165. A straight line from any vertex of a triangle to the middle point of the opposite side is called a *medial*, or a *median*, of the triangle.

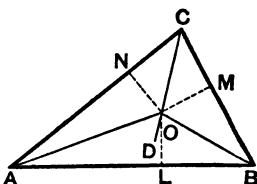
Proposition 41. Theorem.

166. *The bisectors of the three angles of a triangle are concurrent.*

Hyp. Let ABC be a $\triangle$; OA , OB , DC , the bisectors of the $\angle$ s A , B , and C .

To prove that these bisectors meet in a pt.

Proof. Let the bisectors OA , OB meet at O .



Draw the $\perp$ s OL , OM , ON .

Because O is on the bisector of $\angle BAC$,

$$\therefore OL = ON. \quad (160)$$

Because O is on the bisector of $\angle ABC$,

$$\therefore OL = OM. \quad (160)$$

$$\therefore ON = OM. \quad (\text{Ax. 1})$$

$\therefore O$ is on the bisector of the $\angle ACB$,
being equally distant from its sides [160 (2)],

that is, O must lie on the bisector CD .

$\therefore$ the bisectors of the three $\angle$ s meet at the point O .

Q.E.D.

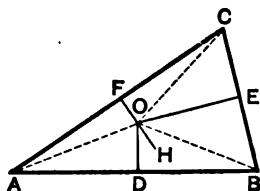
167. COR. *The point O is equally distant from the three sides of the triangle.*

Proposition 42. Theorem.

168. *The perpendicular bisectors of the three sides of a triangle are concurrent.*

Hyp. Let ABC be a $\triangle$; D, E, F , the middle pts. of its sides; and DO, EO, FH , the $\perp$ s erected at D, E, F .

To prove that these $\perp$ s meet in a pt.



Proof. The two $\perp$ s OD, OE , since they cannot be $\parallel$, will meet at the pt. O .

Join OA, OB, OC .

Because O is in the $\perp$ bisector OD ,

$$\therefore OA = OB. \quad (66)$$

Because O is in the $\perp$ bisector OE ,

$$\therefore OB = OC. \quad (66)$$

$$\therefore OA = OC. \quad (\text{Ax. } 1)$$

$$\therefore O \text{ is in the } \perp \text{ bisector of } AC; \quad (67)$$

that is, O must lie on the $\perp$ bisector HF .

$\therefore$ the three $\perp$ bisectors meet at the pt. O . **Q.E.D.**

169. COR. *The point of intersection of the perpendicular bisectors, is equally distant from the three vertices of the triangle.*

EXERCISES.

1. If the triangle in Prop. 41 is equilateral, find the value of the angle AOB .

2. If the same triangle is isosceles, and the angle C is three times as great as either of the angles A and B , find the value of the angle AOB .

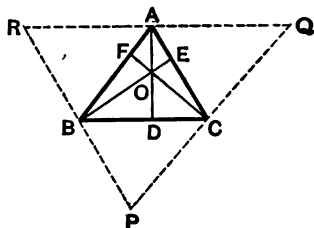
Proposition 43. Theorem.

170. *The perpendiculars from the vertices of a triangle to the opposite sides are concurrent.*

Hyp. Let ABC be a $\triangle$, and AD, BE, CF , the three $\perp$ s from A, B, C to the opp. sides.

To prove that these $\perp$ s meet in a pt.

Proof. Through A, B, C , draw $RQ, RP, PQ \parallel$ respectively to BC, AC, AB .



Then, since $ABPC$ is a $\square$, (124)

$\therefore BP = AC$. (129)

And since $ARBC$ is a $\square$, (124)

$\therefore RB = AC$. (129)

$\therefore RB = BP$. (Ax. 1)

Similarly, $RA = AQ$, and $PC = CQ$;

that is, A, B, C , are the mid. pts. of QR, RP, PQ .

Since AC is $\parallel$ to PR , and BE is $\perp$ to AC ,

$\therefore BE$ is also $\perp$ to PR .

A line $\perp$ to one of two $\parallel$ s is $\perp$ to the other (71).

Similarly, AD and CF are $\perp$ to RQ and PQ .

$\therefore$ these three $\perp$ s meet in a pt.,

being the $\perp$ bisectors of the three sides of the $\triangle PQR$ (168). Q.E.D.

171. DEF. The intersection of the perpendiculars from the vertices of a triangle to the opposite sides is called its *orthocentre*.

The triangle formed by joining the feet of the perpendiculars is called the *pedal* or *orthocentric triangle*.

Proposition 44. Theorem.

172. *The three medial lines of a triangle are concurrent in a point of trisection,* the greater segment in each being towards the angular point.*

Hyp. Let ABC be a $\triangle$; D, E, F , the mid. pts. of BC, AC, AB ; and AD, BE, CF , the three medial lines.

To prove that these lines meet in the pt. that trisects them.

Proof. Let the two medials BE and CF meet in O . Bisect BO in H , and CO in K .

Join HK, KE, EF, FH .

In the $\triangle BOC$, because H and K are the mid. pts. of BO and CO ,

$$\therefore HK \text{ is } \parallel \text{ and } = \frac{1}{2} BC. \quad (155)$$

Also, in the $\triangle ABC$, because E and F are the mid. pts. of AC and AB ,

$$\therefore EF \text{ is } \parallel \text{ and } = \frac{1}{2} BC. \quad (155)$$

$$\therefore EFHK \text{ is a } \square,$$

having two opp. sides $=$ and $\parallel$ (133).

$$\therefore O \text{ is the mid. pt. of } EH \text{ and } FK.$$

The diagonals of a $\square$ bisect each other (134).

$$\therefore EO = OH = HB, \text{ and } FO = OK = KC;$$

or

$$EO = \frac{1}{3} EB, \text{ and } FO = \frac{1}{3} FC.$$

That is, the medial BE cuts the medial CF at a pt. O , one-third the way from F to C .

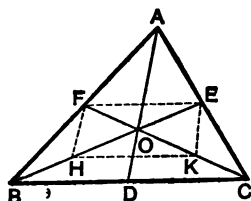
In the same way it may be proved that the medial AD cuts the medial CF at a pt. one-third the way from F to C ; that is, at the same pt. O .

$\therefore$ The three medials are concurrent in the point that trisects them.

Q.E.D.

NOTE.—The point of intersection of the three medians of a triangle is called the *centroid*.

* When a line is divided into three equal parts it is said to be *trisected*.



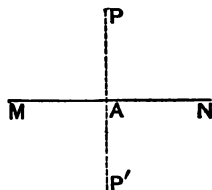
SYMMETRY.

Symmetry with respect to an axis.

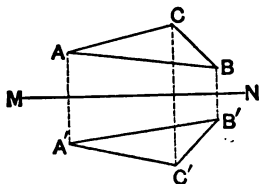
173. Two points are said to be *symmetrical* with respect to a straight line, when the straight line bisects at right angles the straight line joining the two points.

Thus, the two points P and P' are symmetrical with respect to the line MN , if MN bisects PP' at right angles.

The straight line MN is called the *axis of symmetry*.



If we take the plane containing the pt. P , and turn it about the axis MN , until the upper part is brought down on the part below MN , the line AP will take the direction AP' , and the point P will coincide with the point P' . Thus, when two points are symmetrical with respect to an axis, if one of the parts of the plane be revolved about the axis to bring it down on the other part, the symmetrical points coincide.



174. Two figures are said to be *symmetrical* with respect to an axis, when every point in one figure has its symmetrical point in the other.

Thus, the figures ABC , $A'B'C'$ are symmetrical with respect to the axis MN , if every point in the figure ABC has a symmetrical point in $A'B'C'$ with respect to MN .

In all cases, two figures that are symmetrical with re-

spect to an axis, can be applied one to the other, by revolving either about the axis; consequently they are equal.

The corresponding symmetrical lines of symmetrical figures are called *homologous* lines. Thus, in the symmetrical figures, ABC , $A'B'C'$, the homologous lines are AB and $A'B'$, BC and $B'C'$, AC and $A'C'$.

Symmetry with respect to a point.

175. Two points are said to be *symmetrical* with respect to a third point, when this third point bisects the straight line joining the two points.

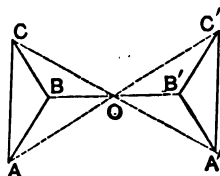
Thus, P and P' are symmetrical with respect to A , if the straight line PP' is bisected at A .



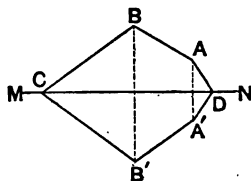
The point A is called the *centre of symmetry*.

176. Two figures are said to be symmetrical with respect to a centre, when every point in one figure has its symmetrical point in the other.

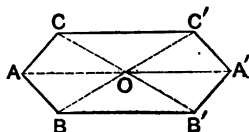
Thus, the figures ABC , $A'B'C'$ are symmetrical with respect to the centre O , if every point in the figure ABC has a symmetrical point in $A'B'C'$.



177. A figure is symmetrical with respect to an axis, when it can be divided by that axis into two figures symmetrical with respect to the axis.

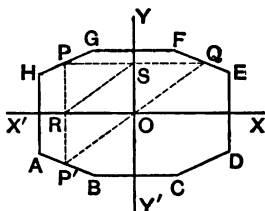


A figure is symmetrical with respect to a centre, when every straight line drawn through that centre cuts the figure in two points symmetrical with respect to this centre.



Proposition 45. Theorem.

178. *If a figure is symmetrical with respect to two axes at right angles to each other, it is also symmetrical with respect to their intersection as a centre.*



Hyp. Let the figure ABCDEFGH be symmetrical with respect to the two $\perp$ axes XX', YY', which intersect at O.

To prove that O is the centre of symmetry of the figure.

Proof. Let P be any pt. in the perimeter of the figure.

Draw $PRP' \perp$ to XX', and $PSQ \perp$ to YY'.

Join RS, OP', and OQ.

Then, because the figure is symmetrical with respect to XX',

$$\therefore PR = P'R.$$

And since

$$PR = OS,$$

$$\therefore P'R = OS.$$

$$\therefore RP'OS \text{ is a } \square,$$

having two opp. sides = and $\parallel$ (133).

$$\therefore RS \text{ is } = \text{ and } \parallel \text{ to } P'O,$$

being opp. sides of a $\square$ (124).

Also, since the figure is symmetrical with respect to YY',

$$\therefore PS = SQ.$$

And since $PS = OR$, $\therefore SQ = OR$.

$$\therefore SROQ \text{ is a } \square; \therefore RS \text{ is } = \text{ and } \parallel \text{ to } OQ.$$

Because both $P'O$ and OQ are = and $\parallel$ to RS ,

$\therefore$ the points P' , O , Q , are in the same st. line which is bisected at O .

$\therefore$ any st. line $P'OQ$, drawn through O , is bisected at O .

$\therefore$ the figure is symmetrical with respect to O as a centre (177).

Q.E.D.

EXERCISES.

The following theorems are given for the exercise of the student, that he may work out his own demonstrations. The student can make no solid acquisitions in geometry, without frequent practice in the application of the principles he has acquired. He should not merely learn the demonstration of a number of theorems, but he should acquire the power of grasping and demonstrating geometric theorems for himself, and this power can never be gained by memorizing demonstrations. He should understand that the ability to investigate, to reason for himself, is the chief object for the attainment of which he should strive. Diligent application, systematic practice in devising proofs of new propositions, is indispensable.

In the process of finding a demonstration the student should first construct a diagram, and state the hypothesis, including in the statement not only what the theorem says, but what it implies. He should also examine the conclusion, and see what it says and what it implies, and discover the relation between the hypothesis and the conclusion. A correct diagram is most useful in suggesting the steps by which a theorem is to be demonstrated. If the student will ask himself why he takes any particular step, he may avoid the habit of random guessing, and with more certainty discover the correct and direct process for effecting the demonstration. Sometimes it will be necessary to draw additional lines in the diagram, and to call to mind the different theorems which apply to the figure thus formed.

The demonstration must be framed in the simplest manner, but without omitting any logical step. This is a matter of practice, in which no general rule can be given.

The student should express each step of the demonstration completely and fully. The most common fault is that of passing over steps in the demonstration because the conclusion seems to be obvious.

It is often the case that the clearness of intuition acquired by a practised geometrician will make him impatient of the successive steps of detailed reasoning, and he will be eager to conduct his pupil to the desired end by a shorter and an easier road. But it is a great misfortune to the learner if he is deprived of that useful discipline in geometric reasoning which, however tedious it may seem in its application to short and easy propositions, is indispensable to the investigation of more lengthy and difficult ones.

One of the great objects of the study of geometry is to cultivate the habit of examining the logical foundations of those conclusions which are accepted without critical examination. The feeling of security that a conclusion is right before its foundation has been examined is a most fruitful source of erroneous opinions, and the person who neglects the habit of inquiring into what appears obvious is liable to pass over things which, had they been carefully examined, would have changed the conclusion.*

1. If the angles ABC and ACB at the base of an isosceles triangle be bisected by the lines BD , CD , show that DBC will be an isosceles triangle.

2. BAC is a triangle having the angle B double the angle A . If BD bisects the angle B and meets AC at D , show that BD is equal to AD .

3. In the triangle ABC , the angle $A = 50^\circ$, the angle $B = 70^\circ$. What angle will the bisectors of these two angles make with each other?

4. In the preceding triangle, what will be the values of the three exterior angles?

5. A given angle BAC is bisected; if CA is produced to G , and the angle BAG bisected, prove that the two bisecting lines are at right angles to each other.

6. ACB , ADB are two triangles on the same side of AB , such that $AC = BD$, and $AD = BC$, and AD and BC intersect at O : prove that the triangle AOB is isosceles.

7. ABC is a triangle, and the angle A is bisected by a line which meets BC at D : show that BA is greater than BD , and CA is greater than CD .

8. If one angle of a triangle is equal to the sum of the other two, the triangle can be divided into two isosceles triangles.

9. If the angle C of a triangle is equal to the sum of the angles A and B, the side AB is equal to twice the line joining C to the middle point of AB.

10. A line bisects the angle A of a triangle ABC; from B a perpendicular is drawn to this bisecting line, meeting it at D, and BD is produced to meet AC or AC produced at E: show that $BD = DE$.

11. A line is drawn terminated by two parallel lines; through its middle point any line is drawn and terminated by the parallel lines. Show that the second line is bisected at the middle point of the first.

12. If through any point equidistant from two parallel lines, two lines be drawn cutting the parallel lines, they will intercept equal portions of these parallel lines.

13. If the line bisecting the exterior angle of a triangle be parallel to the base, show that the triangle is isosceles.

14. If a line be drawn bisecting one of the angles of a triangle to meet the opposite side, the lines drawn from the point of intersection parallel to the other sides, and terminated by these sides, will be equal.

Let ABC be the $\triangle$; let a line be drawn bisecting $\angle A$ and meeting BC at D, etc.

15. The side BC of a triangle ABC is produced to a point D; the angle ACB is bisected by the line CE which meets AB at E. A line is drawn through E parallel to BC, meeting AC at F, and the line bisecting the exterior angle ACD at G. Show that $EF = FG$.

16. A line drawn at right angles to BC, the base of an isosceles triangle ABC, cuts the side AB at D and CA produced at E: show that AED is an isosceles triangle.

From A draw a line bisecting $\angle BAC$, and meeting BC at F, etc.

17. If the lines bisecting the angles at the base of an isosceles triangle be produced to meet, they will contain an angle equal to an exterior angle of the triangle.

18. A is the vertex of an isosceles triangle ABC, and BA is produced to D, so that $AD = BA$; and DC is drawn: show that BCD is a right angle.

19. ABC is a triangle, and the exterior angles at B and C are bisected by the lines BD, CD respectively, meeting at D: show that the angle BDC together with half the angle BAC make up a right angle.

20. In the triangle ABC the side BC is bisected at E, and AB at G; AE is produced to F so that $EF = AE$, and CG is produced to H so that $GH = CG$: show that FB and HB are in one straight line.

21. Lines are drawn through the extremities of the base of an isosceles triangle, making angles with it on the side remote from the vertex, each equal to one-third of one of the equal angles of the triangle and meeting the sides produced: show that the three triangles thus formed are isosceles.

22. AEB, CED are two lines intersecting at E; lines AC, DB are drawn forming two triangles ACE, BED; the angles ACE, DBE are bisected by the lines CF, BF, meeting at F. Show that the angle CFB is equal to half the sum of the angles EAC, EDB.

$$\angle AEC = \angle ECB + \angle EBC, \text{ etc.}$$

23. If a quadrilateral have two of its opposite sides parallel, and the other two equal but not parallel, any two of its opposite angles are together equal to two right angles.

24. On the sides AB, BC, and CD of a parallelogram ABCD three equilateral triangles are described, that on BC towards the same parts as the parallelogram, and those on AB, CD towards the opposite parts: show that the distances of the vertices of the triangles on AB, CD from that

on BC are respectively equal to the two diagonals of the parallelogram.

25. A, B, C are three points in a straight line, such that $AB = BC$: show that the sum of the perpendiculars from A and C on any straight line which does not pass between A and C is double the perpendicular from B on the same line.

26. In a right triangle, show that the medial drawn from the right angle is equal to half the hypotenuse.

27. If P and Q are the feet of the perpendiculars from A on the lines bisecting the angles B and C of a triangle ABC ; show that PQ is parallel to BC .

Produce AP, AQ to meet BC in D, E , etc.

28. AD, BE, CF , the perpendiculars from the vertices of a triangle ABC , intersect in O : prove that the angle $BOF =$ the angle BAC , the angle $FOA =$ the angle ABC , and the angle $BOD =$ the angle BCA .

29. What is the magnitude of each angle of the following regular figures? a pentagon, an octagon, a decagon.

30. From the angle A of a triangle ABC a perpendicular is drawn to the opposite side, meeting it, produced if necessary, at D ; from the angle B a perpendicular is drawn to the opposite side, meeting it, produced if necessary, at E : show that the lines which join D and E to the middle point of AB are equal.

31. From the angles at the base of a triangle perpendiculars are drawn to the opposite sides, produced if necessary: show that the line joining the points of intersection will be bisected by a perpendicular drawn to it from the middle point of the base.

32. The lines which join the middle points of adjacent sides of any quadrilateral, form a parallelogram.

33. The sides AB, AC of the triangle ABC are bisected in D and E respectively; BE and CD are produced to F

and G, so that $EF = BE$ and $DG = DC$: prove that F, A, G are collinear.

34. ABCDE is a regular pentagon; AC, AD are joined: prove that each of the angles ACD, ADC is double the angle CAD.

Draw $DF \parallel$ to AE , F being in AC, etc.

35. Two medians of a triangle are equal: prove (without assuming that they trisect each other) that the triangle is isosceles.

36. ABCD, BACE are parallelograms on the same base AB, such that the diagonals BD, AE are equal: prove that the other diagonals AC, BC are equal.

37. ABCD is a parallelogram; E is the middle point of BC; AB and DE produced meet in F: prove that the triangle DBF is half the parallelogram ABCD.

38. If two right triangles ABC, ABD be on the same hypotenuse AB, and the vertices C and D be joined, the pair of angles subtended by any side of the quadrilateral thus formed are equal.

39. The angles made with the base of an isosceles triangle by perpendiculars from its extremities on the equal sides are each equal to half the vertical angle.

40. The angle included between the internal bisector of one base angle of a triangle and the external bisector of the other base angle is equal to half the vertical angle.

41. The sum of the distances of any point in the base of an isosceles triangle from the equal sides is equal to the distance of either extremity of the base from the opposite side.

42. The sum of the perpendiculars from any point in the interior of an equilateral triangle is equal to the perpendicular from any vertex on the opposite side.

43. AD and BC are two parallel lines cut obliquely by AB, and perpendicularly by AC; and between these lines we draw BED, cutting AC in E, such that $ED = 2AB$: prove that the angle DBC is one-third of ABC.

44. If O be the point of concurrence of the bisectors of

the angles of the triangle ABC, and if AO produced meet BC in D, and from O, OE be drawn perpendicular to BC; prove that the angle BOD = the angle COE.

45. The bisectors of the external angles of a quadrilateral form a circumscribed quadrilateral the sum of whose opposite angles is equal to two right angles.

46. The locus of a point equidistant from two given intersecting lines is two lines at right angles to each other.

Bisect the adj. $\angle$ s between the given lines, etc.

47. Any line is drawn cutting two fixed intersecting lines, and the two angles on the same side of it are bisected: find the locus of the point of intersection of the bisecting lines.

48. The leaf of a book is turned down so that the corner always lies on the same line of printing: find the locus of the foot of the perpendicular from the corner to the crease.

49. ABC is a triangle, D is the middle point of BC, E the middle point of AD; let BE produced meet AC in F: prove that AC is trisected in F.

Draw DG $\parallel$ to BF meeting AC in G, etc.

50. D, E, F are the middle points of the sides of a triangle, N is the foot of the perpendicular from the angle opposite D to the side which D bisects: prove that $\angle EDF = \angle ENF$.

Let D, E, F lie in BC, CA, AB respectively; ED is $\parallel$ to AB, and DF $\parallel$ to AC, etc.

51. If A, B, C denote the angles of a Δ , prove that $\frac{1}{2}(A+B)$, $\frac{1}{2}(B+C)$, $\frac{1}{2}(C+A)$ will be the angles of a Δ formed by any side and the bisectors of the external angles between that side and the other sides produced.

52. If the exterior angles of a Δ be bisected, the three external Δ s formed on the sides of the original Δ are mutually equiangular.

53. If a hexagon have its opposite sides equal and parallel, the three straight lines joining the opposite angles are concurrent.

54. Show that a parallelogram is symmetrical with respect to its centre.

Book II.

THE CIRCLE.

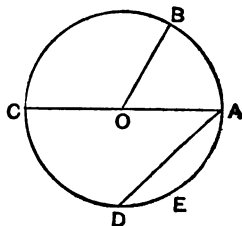
DEFINITIONS.

179. A *circle* is a plane figure bounded by a curved line called the *circumference*, every point of which is equally distant from a point within called the *centre*.

A *radius* is a straight line drawn from the centre to the circumference.

A *diameter* is a straight line drawn through the centre, and terminated both ways by the circumference.

Thus, in the figure, ABCDE is the circumference; the space included within the circumference is the circle; O is the centre; OA, OB, OC, are radii; AOC is a diameter.



From the definition of a circle, it is evident that all its radii are equal; and also that all its diameters are equal, and each double the radius.

180. An *arc* of a circle is any part of the circumference, as AED.

A *semi-circumference* is an arc equal to one-half the circumference.

181. A *chord* is the straight line which joins any two points on the circumference,* as AD.

* The arc is sometimes said to be *subtended* by its chord.

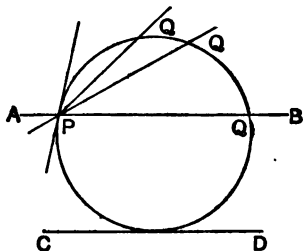
Chords of a circle are said to be *equally distant* from the centre, when the perpendiculars drawn to them from the centre are equal. The chord on which the greater perpendicular falls is said to be *further from the centre*.

NOTE.—Every chord subtends two arcs whose sum is the circumference. A chord which does not pass through the centre, subtends two unequal arcs. Thus, AD subtends both the arc AED and the arc ABCD; of these, the greater is called the *major arc*, and the less the *minor arc*. Thus, the major arc is *greater*, and the minor arc *less* than the semi-circumference.

The major and minor arcs, into which a circumference is divided by a chord, are said to be *conjugate* to each other. When an arc and its chord are spoken of, the minor arc is always meant, unless otherwise stated.

182. A *secant* of a circle is a straight line of indefinite length which cuts the circumference in two points, as AB.

A *tangent* to a circle is any straight line which meets the circumference, but, being produced, does not cut it, as CD. Such a line is said to *touch* the circle at a point, and the point is called the *point of contact*, or *point of tangency*.

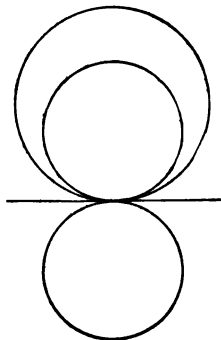


A secant may be considered as a chord produced, and a chord may be considered as the part of the secant that is within the circle.

If a secant, which cuts a circle at the points P and Q, be gradually turned about P as fixed, the point Q will ultimately approach the fixed point P. When the secant PQB reaches this limiting position, it becomes the *tangent* to the circle at the point P.

183. Two circles are tangent to each other when they are tangent to the same straight line at the same point.

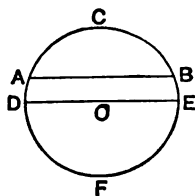
They are said to have *internal contact* or *external contact*, according as one circle is entirely *within* or entirely *without* the other.



Circles that have the same centre are said to be *concentric*.

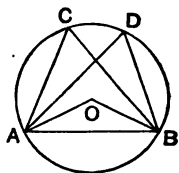
184. A *segment* of a circle is the figure included between a chord and its arc, as ACB, or AFB.

A segment is called a *major* or *minor* segment, according as its arc is a major or minor arc. The chord of the segment is sometimes called the *base* of the segment.



A *semicircle* is the segment included between a diameter and a semi-circumference, as DFE.

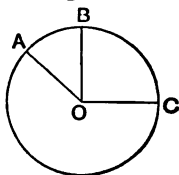
185. A straight line is *inscribed* in a circle when its extremities are on the circumference, as AC.



186. An *inscribed angle* is one whose vertex is in the circumference, and whose sides are chords of the circle, as $\angle ACB$. The $\angle AOB$, whose vertex is at the centre O, is called *the angle at the centre*.

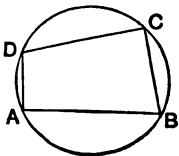
187. An *angle in a segment* is the angle formed by two straight lines drawn from any point in the arc of the segment to the extremities of the base of the segment, as $\angle ACB$, or $\angle ADB$.

188. A *sector* of a circle is the figure bounded by two radii and the arc intercepted between them, as AOB, or BOC.

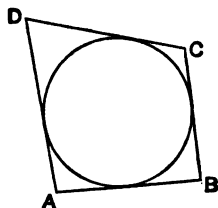


189. A polygon is said to be *inscribed* in a circle, when all its vertices are on the circumference, as ABCD.

190. A circle is said to be *circumscribed about a polygon*, when the circumference passes through each vertex of the polygon.



191. A polygon is said to be *circumscribed about a circle*, when each of its sides is tangent to the circumference, as ABCD. The circle is then said to be *inscribed in the polygon*.



192. From the definition of a circle it follows:

(1) *The distance of a point from the centre of a circle is less than, equal to, or greater than the radius, according as the point is within, on, or without the circumference.*

Hence (157), *the locus of a point, which is always at a given distance from a given point, is the circumference of a circle, of which the given point is the centre, and the given distance is the radius.*

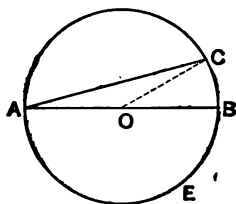
(2) *Circles of equal radii are identically equal.* For, if one circle be applied to the other so that their centres coincide, their circumferences will coincide, since all the points of both are at the same distance from the centre (179).

(3) *A straight line cannot meet the circumference of a circle in more than two points.* For, if it could meet it in three points, these three points would be equally distant from the centre (179). There would then be three equal straight lines drawn from the same point to the same straight line, which is impossible (64).

ARCS AND CHORDS.

Proposition 1. Theorem.

193. *Every diameter divides the circle and the circumference into two equal parts, and is greater than any other chord.*



Hyp. Let AB be the diameter of the $\odot$ ACB, and AC any chord not passing through the centre.

To prove that AB bisects the $\odot$ and the $\odot$ ce, and that $AB > AC$.

Proof. Join OC. Then let the plane containing the arc ACB be turned about AB as an axis until it falls on the plane of AEB.

The arc ACB will coincide with the arc AEB, since every point in ACB is at the same distance from the centre O as every point in AEB. (179)

Hence the arc ACB = the arc AEB,
and the surface ACB = the surface AEB. (29)

Also, $AC < AO + OC$,

i.e., $AO + OC > AC$.

But $AO + OC = AO + OB$,

since OC and OB are radii of the same $\odot$ (179).

$\therefore AO + OB > AC$, or $AB > AC$. Q.E.D.

Proposition 2. Theorem.

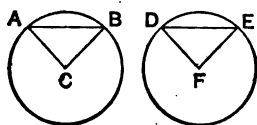
194. *In the same circle, or in equal circles, equal arcs have equal chords, and subtend equal angles at the centre.*

Hyp. Let ABC, DEF be equal $\odot$ s, and AB, DE equal arcs.

To prove

chord AB = chord DE,

and $\angle ACB = \angle DFE$.



Proof. Apply the $\odot$ ABC to the $\odot$ DEF, so that the centre C may fall on the centre F, and the radius CA on the radius FD.

Then, because the $\odot$ s are equal, (Hyp.)

$\therefore$ the pt. A will fall on the pt. D,

and their $\odot$ ces will coincide,

since all the pts. of both are at the same distance from the centre (179).

Because arc AB = arc DE, (Hyp.)

$\therefore$ the pt. B will fall on the pt. E,

and the chord AB will coincide with chord DE. (Ax. 11)

$\therefore$ chord AB = chord DE,

and $\angle ACB = \angle DFE$. Q. E. D.

NOTE.—A line, straight or curved, is said to *subtend* a certain angle from a certain point when the lines drawn from the point to the ends of the line form that angle. Thus, the chord AB or the arc AB subtends the angle ACB from the point C.

195. Cor. *Sectors of equal arcs in the same or equal circles are equal.*

Proposition 3. Theorem.

196. *In the same circle, or in equal circles, equal angles at the centre intercept equal chords and equal arcs on the circumference.*

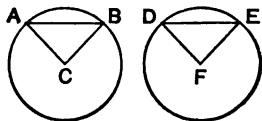
Hyp. Let ABC , DEF be equal $\odot$ s, and let $\angle C = \angle F$.

To prove

chord $AB =$ chord DE ,

and arc $AB =$ arc DE .

Proof. Apply the $\odot ABC$ to the $\odot DEF$, so that the centre C may fall on the centre F , and the radius CA on the radius FD .



Then, because the $\odot$ s are equal, (Hyp.)

$\therefore$ the pt. A will fall on the pt. D ,

and their $\odot$ es will coincide. (179)

Because $\angle C = \angle F$, (Hyp.)

$\therefore$ CB must fall on FE ;

and since $CB = FE$, (Hyp.)

$\therefore$ the pt. B will fall on the pt. E .

Now since A coincides with D , and B with E , and the $\odot$ ce of the $\odot ABC$ with the $\odot$ ce of the $\odot DEF$,

$\therefore$ chord AB must coincide with chord DE , (Ax. 11)

and the arc AB must coincide with the arc DE .

$\therefore$ chord $AB =$ chord DE ,

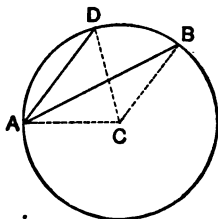
and arc $AB =$ arc DE . Q.E.D.

197. COR. 1. *Sectors of equal angles at the centre in the same or equal circles are equal.*

198. COR. 2. *In the same or equal circles equal chords subtend equal arcs and equal angles at the centre.*

Proposition 4. Theorem.

199. *In the same circle, or in equal circles, of two unequal minor arcs, the greater is subtended by the greater chord.*



Hyp. In the $\odot ABC$, let

arc $AB >$ arc AD .

To prove chord $AB >$ chord AD .

Proof. Draw the radii CA, CD, CB .

Because the arc $AB >$ the arc AD , (Hyp.)

$\therefore$ the pt. D must be between the pts. A and B ,

and $\therefore \angle ACB > \angle ACD$.

Then in the $\triangle s ABC, ADC$, we have

$CB = CD, CA = CA,$

$\angle ACB > \angle ACD$.

$\therefore$ chord $AB >$ chord AD ,

since the $\triangle s$ have two sides equal, each to each, and the included $\angle s$ unequal (119). Q.E.D.

200. COR. Conversely: If the chord $AB >$ the chord AD , the arc $AB >$ the arc AD . For, if the arc AB were equal to the arc AD , the chord AB would be equal to the chord AD (194); and if the arc AB were less than the arc AD , the chord AB would be less than the chord AD (199). Then, since the arc AB cannot be equal to, nor less than the arc AD , it must be greater.

Hence, *the greater chord subtends the greater minor arc.*

Proposition 5. Theorem.

201. *The radius which is perpendicular to a chord bisects the chord and the arc which it subtends.*

Hyp. In the $\odot ABC$ let the radius OD be $\perp$ to the chord AB at E .

To prove $AE = EB$,

and $\text{arc } AD = \text{arc } DB$.

Proof. Join OA , OB .

Then, since $OA = OB$, (Radii)

OE is common,

and $\text{rt. } \angle AEO = \text{rt. } \angle BEO$,

$\therefore \triangle AOE = \triangle BOE$.

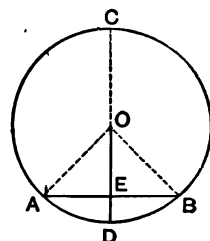
Two rt. $\triangle$ s are equal if they have the hypotenuse and a side equal, each to each (110).

$\therefore AE = EB$,

and $\angle AOE = \angle BOE$.

$\therefore \text{arc } AD = \text{arc } DB$,

since equal $\angle$ s at the centre intercept equal arcs on the $\odot$ (196). Q.E.D.



202. COR. 1. *The perpendicular bisector of a chord passes through the centre of the circle, and bisects both the arc and the angle at the centre subtended by the chord.*

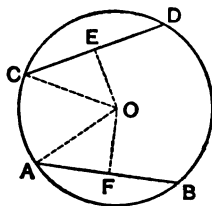
203. COR. 2. *A radius which bisects a chord is perpendicular to the chord and bisects the subtended arc.*

204. COR. 3. *A radius which bisects an arc is perpendicular to the chord of that arc at its middle point, and bisects the angle at the centre which the arc subtends.*

205. COR. 4. *The locus of the middle point of parallel chords in a circle is the diameter perpendicular to these chords.*

Proposition 6. Theorem.

206. *In the same circle, or in equal circles, equal chords are equally distant from the centre; and conversely, chords which are equally distant from the centre are equal to one another.*



(1) *Hyp.* Let AB, CD be equal chords in the $\odot ABDC$, and let OF, OE be $\perp$ to AB, CD .

To prove $OF = OE$.

Proof. Join OA, OC .

Then, since OF, OE are $\perp$ to AB, CD ,

$\therefore AB, CD$ are bisected at F, E . (201)

But $AB = CD$. (Hyp.)

$\therefore AF = CE$. (Ax. 7)

Also, $AO = CO$. (Radii)

$\therefore$ rt. $\triangle AOF =$ rt. $\triangle COE$. (110)

$\therefore OF = OE$.

(2) *Hyp.* Let the $\perp$ s OF, OE be equal.

To prove $AB = CD$.

Proof. Since $OF = OE$, (Hyp.)

and $OA = OC$, (Radii)

$\therefore$ rt. $\triangle AOF =$ rt. $\triangle COE$. (110)

$\therefore AF = CE$,

and $\therefore AB = CD$. (201 and Ax. 6)

Q. E. D.

Proposition 7. Theorem.

207. *In the same circle, or in equal circles, of two unequal chords, the less is at the greater distance from the centre.*

Hyp. In the $\odot$ ABD, let the chord $CD <$ the chord AB , and let the $\perp$ s OE, OF be drawn from the centre O to CD, AB .

To prove $OE > OF$.

Proof.

Since chord $AB >$ chord CD ,

$\therefore$ arc $AB >$ arc CD : (200)

On arc AGB take arc $AG =$ arc CD .

Draw the chord AG , and the $\perp$ OH .

Since arc $AG =$ arc CD , $\therefore$ chord $AG =$ chord CD .

Equal arcs have equal chords (194).

And $\therefore OH = OE$.

Equal chords are equally distant from the centre (206).

Because arc $AG <$ arc AB , the pt. G must fall within the arc AB .

$\therefore \perp OH$ will cut the chord AB at some pt. K .

Now, $OH > OK$.

The whole is $>$ any of its parts (Ax. 8).

And $OK > OF$.

The $\perp$ is the shortest distance from a pt. to a line (58).

$\therefore$ a fortiori, $OH > OF$.

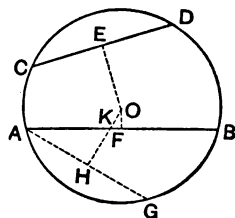
$\therefore OE > OF$.

Q.E.D

208. COR. *Conversely, of two chords unequally distant from the centre, the one which is at the greater distance is the less.*

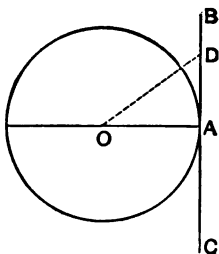
EXERCISE.

If two chords of a circle cut each other, and make equal angles with the straight line which joins their point of intersection to the centre, prove that the chords are equal.



Proposition 8. Theorem.

209. *A straight line perpendicular to a radius at its extremity is a tangent to the circle.*



Hyp. Let O be the centre of a $\odot$, OA the radius, and BC a line $\perp$ to OA at A .

To prove BC tangent to the $\odot$.

Proof. In BC take any pt. D , other than A ; join OD .

Then, since OA is $\perp$ to BC , (Hyp.)

$$\therefore OA < OD.$$

The $\perp$ is the shortest distance from a pt. to a line (58).

$\therefore$ the pt. D is without the circle. [192. (1)]

$\therefore BC$ has every pt. except A without the $\odot$,
and $\therefore BC$ is a tangent to the $\odot$ at A . (182)

Q. E. D.

210. COR. 1. *Conversely, a tangent to a circle at any point is perpendicular to the radius drawn to that point.*

211. COR. 2. *The perpendicular to a tangent at the point of tangency passes through the centre of the circle.*

212. COR. 3. *The straight line drawn from the centre perpendicular to the tangent meets it in the point of contact.*

213. COR. 4. *Only one tangent can be drawn to a circle at a given point on the circumference.*

Proposition 9. Theorem.

214. *Two parallel lines intercept equal arcs on the circumference.*

There may be three cases.

Hyp. Let AB, CD be the $\parallel$ s.

CASE I. *When AB, CD are secants.*

To prove arc $AC =$ arc BD .

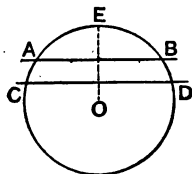
Proof. Draw the radius $OE \perp$ to one of the $\parallel$ s.

It will then be $\perp$ to the other $\parallel$.

A st. line $\perp$ to one of two $\parallel$ s is $\perp$ to the other (71).

$\therefore$ arc $AE =$ arc BE , and arc $CE =$ arc DE , (201)

$\therefore$ arc $AC =$ arc BD . (Ax. 3)



CASE II. *When AB is a tangent and CD is a secant.*

To prove arc $CE =$ arc DE .

Proof. Draw the radius OE to the pt. of contact E .

Then OE is $\perp$ to AB , (210)

and to it $\parallel$ CD . (71)

$\therefore$ arc $CE =$ arc DE . (201)

CASE III. *When AB, CD are tangents.*

To prove

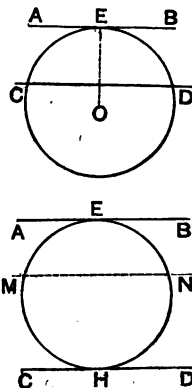
arc $EMH =$ arc ENH .

Proof. Draw the secant $MN \parallel$ to AB .

Then arc $ME =$ arc NE ,
and arc $MH =$ arc NH . } (Case II)

Adding,

arc $EMH =$ arc ENH . Q.E.D.

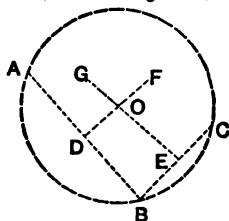


215. COR. 1. *Conversely, if the arcs intercepted by two secants are equal, the secants are parallel.*

216. COR. 2. *The straight line joining the points of contact of two parallel tangents is a diameter.*

Proposition 10. Theorem.

217. *Through three given points not in the same straight line, one circumference, and only one, can be drawn.*



Hyp. Let A, B, C be the three given pts. not in a st. line.

To prove that one $\odot$ ce, and only one, can be drawn through A, B, C.

Proof. Join AB, BC.

Bisect AB, BC by the $\perp$ s DF, EG.

Since AB, BC are not in the same st. line, (Hyp.)

$\therefore$ the $\perp$ s DF, EG must meet at some pt. O. (76)

Because O is in the $\perp$ DF,

$\therefore$ it is equidistant from A and B. (66)

And because O is in the $\perp$ EG,

$\therefore$ it is equidistant from B and C. (66)

Then, because O is equidistant from A, B, C,

$\therefore$ the $\odot$ ce described with centre O and radius OA will pass through A, B, C.

Again, only one $\odot$ ce can be so described.

For if any $\odot$ ce pass through A, B, C, its centre will be at once in the $\perp$ bisectors DF, EG, and $\therefore$ at their pt. of intersection. But two st. lines cannot intersect in more than one pt.

$\therefore$ there is only one $\odot$ ce that can pass through A, B, and C.

Q.E.D.

218. COR. 1. *Two circumferences cannot intersect in more than two points.*

219. COR. 2. *Two circumferences which have three points common coincide.*

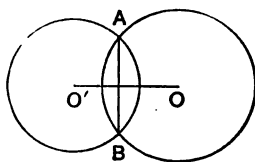
RELATIVE POSITION OF TWO CIRCLES.

Proposition 11. Theorem.

220. *If two circumferences intersect each other, the right line joining their centres bisects their common chord at right angles.*

Hyp. Let O, O' be the centres of two $\odot$ ces which intersect each other; and A, B their pts. of intersection.

To prove that the line OO' bisects AB at rt. $\angle$ s.

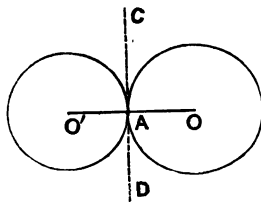


Proof. Because O and O' are each equally distant from A and B , (179)

$\therefore$ the line OO' bisects AB at rt $\angle$ s. (67)

221. COR. 1. *Conversely, the perpendicular bisector of a common chord passes through the centres of both circles.*

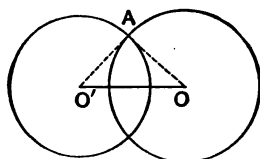
222. COR. 2. If we suppose the circles to be moved so that the point A approaches the line OO' , the pt. B will also approach the line; and since the line OO' is always perpendicular to the middle of AB (220), the two points A and B will ultimately come together on the line OO' , and be united in a single point common to the two circles. The common chord AB will then be a common tangent to the two circumferences at their point of contact.



Hence, *when two circumferences are tangent to each other, their point of contact is in the straight line joining their centres; and the perpendicular at this point is a common tangent to the two circumferences.*

Proposition 12. Theorem.

223. *If two circumferences intersect each other, the distance between their centres is less than the sum and greater than the difference of the radii.*



Hyp. Let the $\odot$ s with centres O, O' intersect at A . Join OO', AO, AO' .

To prove $OO' < OA + AO'$, and $> OA - AO'$.

Proof. In the $\triangle OAO'$,

$$OO' < OA + AO', \text{ and } OO' > OA - AO'.$$

Either side of a $\triangle$ < the sum and > the difference of the other two sides (96). Q.E.D.

224. COR. 1. *If the distance of the centres of two circles is greater than the sum of their radii, they are wholly exterior to each other.*

225. COR. 2. *If the distance of the centres of two circles is equal to the sum of the radii, they are tangent externally.*

226. COR. 3. *If the distance of the centres is less than the sum and greater than the difference of the radii, the circles intersect.*

227. COR. 4. *If the distance of the centres is equal to the difference of the radii, the circles are tangent internally.*

228. COR. 5. *If the distance of the centres is less than the difference of the radii, one circle is wholly within the other.*

SCH. If two circles intersect and the radius of either circle drawn to a point of section touches the other circle, the circles intersect *orthogonally*, i.e., at right angles.

EXERCISES.

1. Through a given point P either inside or outside a given circle whose centre is O , two straight lines PAB , PCD are drawn making equal angles with OP , and cutting the circle in A , B , C , D : prove that $AB = CD$, and $PA = PC$.

2. P is a point inside a circle whose centre is O : prove that the chord which is at right angles to OP is the shortest chord that can be drawn through P .

Let APB be $\perp$ to OP , CPD any other chord through P ; draw $OE \perp$ to CD , etc.

3. If two circles cut each other, any two parallel straight lines drawn through the points of intersection to cut the circles are equal.

4. Two circles whose centres are A and B intersect at C ; through C two chords DCE , FCG are drawn equally inclined to AB and terminated by the circles: show that $DE = FG$.

5. Prove that the two tangents drawn to a circle from an external point are equal and equally inclined to the straight line joining the point to the centre of the circle.

6. A is a point outside a given circle whose centre is O ; with centre A and radius AO a circle is described, and with centre O and radius equal to the diameter of the given circle another circle is described, cutting the last in B ; OB is joined, cutting the given circle in E : prove that AE is tangent to the given circle.

THE MEASUREMENT OF ANGLES.

229. To *measure* a quantity is to find how many times it contains another quantity of the same kind taken as a standard of comparison. This standard is called the *unit*.

Thus, if we wish to measure a line, we must take a *unit of length*, and see how many times it is contained in the line to be measured.

The number which shows how many times a quantity contains the unit, is called the *numerical measure* of that quantity.

230. The relative magnitude of two quantities, measured by the number of times which the first contains the second, is called their *ratio*. Thus, the ratio of A to B is

$$\frac{A}{B}, \text{ or } A : B.$$

Since the ratio of two quantities is found by dividing the first by the second, therefore the ratio of two quantities is the number which would express the measure of the first, if the second were taken as unity.

The ratio of two quantities is the same as the ratio of their numerical measures.

Thus, if A contains the unit m 28 times, and B contains it 9 times, we have $\frac{A}{B} = \frac{28m}{9m} = \frac{28}{9}$.

231. When a quantity is contained an exact number of times in two quantities of its kind, it is called their *common measure*.

Two quantities are *commensurable* when they have a common measure.

The ratio of two commensurable quantities can be expressed by a whole number or by a fraction.

Thus, if each of the two lines A and B contains some line C an exact number of times, A  as for example if A contains it 5 times and B contains it 4 times, B  the two lines A and B are commensurable, and the line C is their common measure. Their ratio is expressed by the fraction $\frac{5}{4}$.

If C is not contained an exact number of times in A and B, but if there be a common measure which is contained, say, 25 times in A and 19 times in B, then the ratio of A to B is the fraction $\frac{25}{19}$.

Generally, if there be a common measure which is contained m times in A and n times in B, their ratio will be $\frac{m}{n}$.

232. Two quantities are *incommensurable* when they have no common measure. The ratio of such quantities is called an *incommensurable* ratio. This ratio cannot be exactly expressed in figures; but its numerical value can be obtained approximately as near as we please.

Thus, suppose A and B are two lines whose ratio is $\sqrt{2}$. We cannot find any fraction which is *exactly* equal to $\sqrt{2}$; but by taking a sufficient number of decimals, we may find $\sqrt{2}$ to any required degree of approximation.

Thus, $\sqrt{2} = 1.4142135 \dots$,

and therefore $\sqrt{2} > 1.414213$ and < 1.414214 .

That is, the ratio of A to B lies between $\frac{1414213}{1000000}$ and $\frac{1414214}{1000000}$, and therefore differs from either of these ratios by less than one-millionth. And since the decimals may be continued without end in extracting the square root of 2, it is evident that this ratio can be expressed as a fraction with an error *less than any assignable quantity*.

In general, when A and B are incommensurable, divide B into n equal parts each equal to x , so that $B = nx$, where n is an integer. Also let $A > mx$ but $< (m+1)x$; then

$$\frac{A}{B} > \frac{mx}{nx} \quad \text{and} \quad < \frac{(m+1)x}{nx};$$

that is, $\frac{A}{B}$ lies between $\frac{m}{n}$ and $\frac{m+1}{n}$; so that $\frac{A}{B}$ differs from $\frac{m}{n}$ by a quantity less than $\frac{1}{n}$. And since n can be taken as great as we please, $\frac{1}{n}$ may be made as small as we

please until it becomes less than any assignable value, though it can never reach absolute zero.

We say, therefore, that zero is the *limit* of $\frac{1}{n}$ as n is increased indefinitely.

Hence, two integers can be found whose ratio will express the ratio of two incommensurable quantities to any required degree of accuracy.

233. THEOREM. *Two incommensurable ratios are equal, if their approximate numerical values always remain equal while the common measure is indefinitely diminished.*

Hyp. Let $A : B$ and $A' : B'$ be two incommensurable ratios, whose true values always differ from $\frac{m}{n}$ by less than $\frac{1}{n}$.

To prove $A : B = A' : B'$.

Proof. Since each of the ratios $\frac{A}{B}$, $\frac{A'}{B'}$ differs from $\frac{m}{n}$ by less than $\frac{1}{n}$,

$\therefore$ the difference between $\frac{A}{B}$ and $\frac{A'}{B'}$ must be $< \frac{1}{n}$.

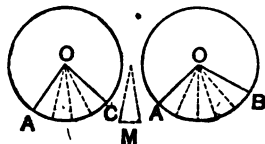
But, by diminishing the common measure, n can be made as great as we please, and therefore $\frac{1}{n}$ may be made as small as we please, i.e., less than any assignable quantity, however small. Hence, the difference between $A : B$ and $A' : B'$ must be less than any assignable quantity, however small.

$\therefore A : B = A' : B'$.

Proposition 13. Theorem.

234. *In the same circle, or in equal circles, angles at the centre are in the same ratio as their intercepted arcs.*

Hyp. Let $\angle AOB$, $\angle AOC$ be any two $\angle$ s at the centres of two equal $\odot$ s, and AB , AC their intercepted arcs.



To prove
$$\frac{\angle AOB}{\angle AOC} = \frac{\text{arc } AB}{\text{arc } AC}.$$

CASE I. *When the arcs are commensurable.*

Proof. Take M , any common measure of AB and AC , and suppose it to be contained five times in AB and four times in AC .

Then
$$\frac{\text{arc } AB}{\text{arc } AC} = \frac{5}{4}. \quad (1)$$

Draw radii to the several pts. of division of the arcs AB , AC , dividing the $\angle AOB$ into five $\angle$ s and $\angle AOC$ into four $\angle$ s.

These $\angle$ s are all equal.

In the same $\odot$, or in equal $\odot$ s, equal arcs subtend equal $\angle$ s at the centre (194).

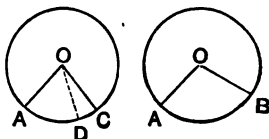
$\therefore \frac{\angle AOB}{\angle AOC} = \frac{5}{4}. \quad (2)$

Therefore, from (1) and (2),

$$\frac{\angle AOB}{\angle AOC} = \frac{\text{arc } AB}{\text{arc } AC}. \quad (\text{Ax. 1})$$

CASE II. *When the arcs are incommensurable.*

Proof. In this case we know (232) that we may always find an arc AD as nearly equal as we please to AC , and such that AB , AD are commensurable.



Join OD ; then

$$\frac{\angle AOB}{\angle AOD} = \frac{\text{arc } AB}{\text{arc } AD}. \quad (\text{Case I})$$

Now, these two ratios being always equal while the common measure is indefinitely diminished, they will be equal when D moves up to and *as nearly as we please* coincides with C. (233)

$$\therefore \frac{\angle AOB}{\angle AOC} = \frac{\text{arc } AB}{\text{arc } AC}. \quad \text{Q.E.D.}$$

235. COR. *In the same circle, or in equal circles, sectors are in the same ratio as their arcs; for sectors are equal when their arcs are equal.* (195)

236. SCH. Since the angle at the centre of a circle, and the arc intercepted by its sides, increase and decrease in the same ratio (234), the numerical measure of the angle is the same as that of the arc. This theorem, being of very frequent use, is expressed briefly by saying that *an angle at the centre is measured by its intercepted arc.** This means, simply, that an angle at the centre is the same part of the whole angular magnitude about the centre that its intercepted arc is of the whole circumference.

237. The circumference is divided, like the angular magnitude about the centre (28), into 360 equal parts called degrees. The degree is divided into 60 equal parts called minutes, and the minute into 60 equal parts called seconds. Hence the unit of angle and the unit of arc are both called a degree. When the angle becomes a right angle (20), the arc becomes a quarter of the circumference, or a *quadrant*. When the angle becomes a straight angle (21), the arc becomes a semi-circumference, and so on. A right angle and a quadrant are both expressed by 90° . Two right angles and a semi-circumference are both expressed by 180° . Four right angles and a circumference are both expressed by 360° .

* Rouché et Comberousse, p. 64.

Proposition 14. Theorem.

238. *An inscribed angle is measured by one-half the arc intercepted between its sides.*

Hyp. Let $\angle BAC$ be an $\angle$ inscribed in the $\odot ABC$ and intercepting the arc BC .

To prove $\angle BAC$ is measured by $\frac{1}{2}$ arc BC .

CASE I. *When the centre O is within the $\angle BAC$.*

Proof. Join AO , produce it to D , and join OB , OC .

In $\triangle AOB$,

$$\angle BOD = \angle OAB + \angle OBA.$$

The ext. $\angle$ of a $\triangle$ equals the sum of the opp. int. $\angle$ s (98).

But, since $OA = OB$, (Radii)

$$\therefore \angle OAB = \angle OBA,$$

being opp. equal sides (111).

$$\therefore \angle BOD = 2 \angle OAB.$$

Similarly, $\angle COD = 2 \angle OAC$.

$$\therefore \text{whole } \angle BOC = 2 \angle BAC. \quad (\text{Ax. 2})$$

But $\angle BOC$ is measured by arc BC .

The $\angle$ at the centre is measured by the intercepted arc (236).

$$\therefore 2 \angle BAC \text{ is measured by arc } BC.$$

$$\therefore \angle BAC \text{ is measured by } \frac{1}{2} \text{ arc } BC.$$

CASE II. *When the centre O is without the $\angle BAC$.*

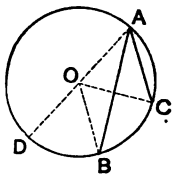
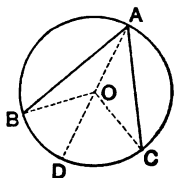
Proof. Join AO , produce it to D , and join OB , OC . Then,

$$\angle DOB = 2 \angle DAB, \text{ (Case I)}$$

$$\text{and } \angle DOC = 2 \angle DAC. \text{ (Case I)}$$

$$\therefore \angle DOC - \angle DOB = 2 \angle DAC - 2 \angle DAB.$$

$$\therefore \angle BOC = 2 \angle BAC.$$

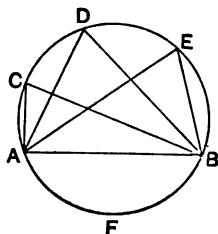


But $\angle BOC$ is measured by arc BC. (236)

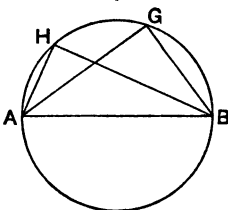
$\therefore \angle BAC$ is measured by $\frac{1}{2}$ arc BC. Q.E.D.

NOTE.—This theorem is equally true when the angle at the centre is greater than two right angles, as the student may show.

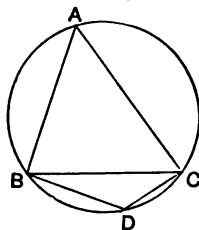
239. COR. 1. *All angles inscribed in the same segment are equal; for each is measured by one-half the same arc AFB.*



240. COR. 2. *Every angle AHB, inscribed in a semicircle, is a right angle; for it is measured by one-half a semi-circumference, or by a quadrant. (237)*



241. COR. 3. *Every angle BAC, inscribed in a segment greater than a semicircle, is an acute angle; for it is measured by one-half the arc BDC, which is less than a quadrant.*



Every angle BDC, inscribed in a segment less than a semicircle, is an obtuse angle; for it is measured by one-half the arc BAC, which is greater than a quadrant.

242. COR. 4. *The opposite angles of an inscribed quadrilateral are supplementary; for the sum of the $\angle$ s A and D is measured by one-half the $\odot$ ce, which is the measure of two right $\angle$ s (237); therefore the $\angle$ s are supplementary. (25)*

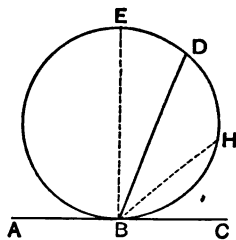
Proposition 15. Theorem.

243. *An angle formed by a tangent and a chord from the point of contact is measured by one-half the intercepted arc.*

Hyp. Let AC be a tangent to the $\odot$ BHE at B, and BD any chord of the $\odot$ from B.

To prove $\angle CBD$ is measured by $\frac{1}{2}$ arc BHD.

Proof. From B draw BE at rt. $\angle$ s to AC.



BE is a diameter of the $\odot$.

The $\perp$ to a tangent at the pt. of contact passes through the centre of the $\odot$ (211).

Then, rt. $\angle CBE$ is measured by $\frac{1}{2}$ semi $\odot$ ce BHE,

and $\angle DBE$ is measured by $\frac{1}{2}$ arc DE. (238)

Therefore, subtracting,

$\angle CBD$ is measured by $\frac{1}{2}$ arc BHD.

Similarly, $\angle ABD$ is measured by $\frac{1}{2}$ arc BED. Q. E. D.

244. SCH. This proposition is a particular case of Prop. 14. Thus, let the side BD remain fixed, while the side BH turns about B, as in (28), until it becomes the tangent BC at the point B. In every position of the chord BH, the inscribed angle HBD is measured by half the intercepted arc HD. Therefore, when the chord BH becomes the tangent BC, the angle CBH is measured by half the arc BHD.

EXERCISES.

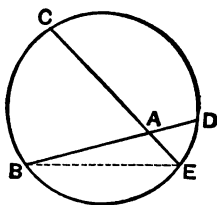
1. If the angle BAC at the circumference of a circle be half that of an equilateral triangle, prove that BC is equal to the radius of the circle.

2. If a hexagon be inscribed in a circle, show that the sum of any three alternate angles is four right angles.

3. If two circles intersect in the points A, B, and any two lines ACD, BFE, be drawn through A and B, cutting one of the circles in the points C, E, and the other in the points D, F, the line CE is parallel to DF.

Proposition 16. Theorem.

245. *An angle formed by two chords which intersect within a circle, is measured by one-half the sum of the arcs intercepted between its sides and between its sides produced.*



Hyp. Let BD, CE be two chords intersecting at A within the $\odot$ BCDE.

To prove $\angle BAC$ is measured by $\frac{1}{2}$ (arc BC + arc DE).

Proof. Join BE.

$$\angle BAC = \angle AEB + \angle ABE.$$

The ext. $\angle$ of a $\triangle$ equals the sum of the opp. int. $\angle$ s (98).

But $\angle AEB$ is measured by $\frac{1}{2}$ arc BC,

and $\angle ABE$ is measured by $\frac{1}{2}$ arc DE. (238)

$\therefore$ adding, $\angle BAC$ is measured by $\frac{1}{2}$ (arc BC + arc DE).

Q. E. D.

EXERCISES.

1. If arc BC = 84° and $\angle CAD$ is a rt. $\angle$, how many degrees are there in the arc DE?

2. The sides of a quadrilateral touch a circle, and the straight lines drawn from the centre of the circle to the vertices cut the circumference in A, B, C, D: show that AC, BD, which intersect inside the circle, are at right angles to each other.

Proposition 17. Theorem.

246. *An angle formed by two secants which intersect without a circle, is measured by one-half the difference of the intercepted arcs.*

Hyp. Let AC, AB be two secants intersecting at A without the $\odot$ BCED.

To prove $\angle A$ is measured by $\frac{1}{2}$ (arc BC - arc DE).

Proof. Join BE.

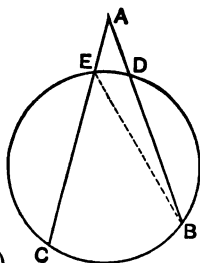
$$\angle BEC = \angle A + \angle B. \quad (98)$$

$$\therefore \angle A = \angle BEC - \angle B.$$

But $\angle BEC$ is measured by $\frac{1}{2}$ arc BC,

and $\angle B$ is measured by $\frac{1}{2}$ arc DE. (238)

$\therefore \angle A$ is measured by $\frac{1}{2}$ (arc BC - arc DE). Q. E. D.



247. SCH. Prop. 14 may be considered as a special case of Props. 16 and 17 by conceiving AB in (245) and (246) to move parallel to its present position until D reaches E. When D reaches E, the arc DE becomes zero, and BAC becomes an inscribed angle, measured by half its intercepted arc.

EXERCISES.

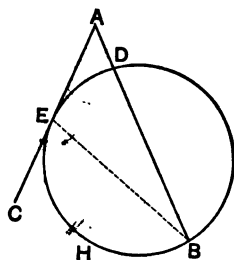
1. If arc BC = 80° and $\angle B = 14^\circ$, find the number of degrees in the angle A.

2. A, B, C are three points on the circumference of a circle, the bisectors of the angles A, B, C meet in D, and AD produced meets the circle in E: prove that ED = EC.

3. If a quadrilateral be described about a circle, the angles at the centre subtended by the opposite sides are supplemental.

Proposition 18. Theorem.

248. *An angle formed by a tangent and a secant intersecting at A is measured by one-half the difference of the intercepted arcs.*



Hyp. Let AC, AB be a tangent and a secant intersecting at A.

To prove $\angle A$ is measured by $\frac{1}{2}$ (arc BHE — arc DE).

Proof. Join BE.

$$\angle BEC = \angle A + \angle B. \quad (98)$$

$$\therefore \angle A = \angle BEC - \angle B.$$

$$\text{But } \angle BEC \text{ is measured by } \frac{1}{2} \text{ arc BHE}, \quad (238)$$

$$\text{and } \angle B \text{ is measured by } \frac{1}{2} \text{ arc DE}. \quad (238)$$

$$\therefore \angle A \text{ is measured by } \frac{1}{2} (\text{arc BHE} - \text{arc DE}).$$

Q.E.D.

249. COR. *The angle formed by two tangents intersecting at A is measured by one-half the difference of the intercepted arcs.*

EXERCISES.

1. Two tangents AB, AC are drawn to a circle; D is any point on the circumference outside the triangle ABC: show that the sum of the angles ABD and ACD is constant.

2. If a variable tangent meets two parallel tangents it subtends a right angle at the centre.

QUADRILATERALS.

Proposition 19. Theorem.

250. *If the opposite angles of a quadrilateral are supplementary, the quadrilateral may be inscribed in a circle.*

Hyp. Let ABCD be a quadrilateral in which $\angle B + \angle D = 2 \text{ rt. } \angle s.$

To prove the pts. A, B, C, D are in the same $\odot$.

Proof. Through the three pts. A, B, C describe a $\odot$.

If this $\odot$ does not pass through D, it will cut AD, or AD produced, at some other pt. than D.

Let E be this pt. Join EC.

Because the quadrilateral ABCE is inscribed in a $\odot$,

$$\therefore \angle ABC + \angle AEC = 2 \text{ rt. } \angle s.$$

The opp. $\angle s$ of an inscribed quad. are supplementary (242).

But $\angle ABC + \angle ADC = 2 \text{ rt. } \angle s.$ (Hyp.)

$$\therefore \angle ABC + \angle AEC = \angle ABC + \angle ADC. \quad (\text{Ax. } 1)$$

$$\therefore \angle AEC = \angle ADC; \quad (\text{Ax. } 3)$$

that is, an ext. $\angle$ of a $\triangle$ = an int. opp. $\angle$, which is impossible. (98)

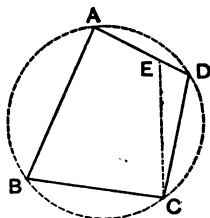
$\therefore$ the circle which passes through A, B, C, must pass through D. Q.E.D.

251. DEF. Points which lie on the circumference of a circle are called *concyclic*.

A *cyclic quadrilateral* is one which is inscribed in a circle.

EXERCISE.

If two opposite sides of a cyclic quadrilateral be produced to meet, and a perpendicular be let fall on the bisector of the angle between them from the point of intersection of the diagonals: prove that this perpendicular will bisect the angle between the diagonals.



Proposition 20. Theorem.

252. *In any quadrilateral circumscribing a circle, the sum of one pair of opposite sides is equal to the sum of the other pair.*

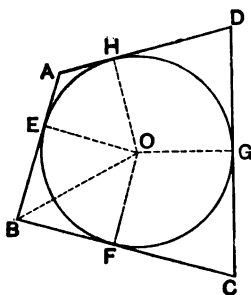
Hyp. Let ABCD be a quadrilateral circumscribing a $\odot$.

To prove $AB + CD = AD + BC$.

Proof. From the centre O draw the radii to the pts. of contact E, F, G, H, and draw OB.

Then rt. $\triangle OBE =$ rt. $\triangle OBF$, (110)

$\therefore EB = FB$.



Similarly, $EA = HA$, $GD = HD$, $GC = FC$.

Adding these four equations, we have

$$EB + EA + GD + GC = FB + HA + HD + FC,$$

or

$$AB + CD = BC + AD.$$

Q.E.D.

EXERCISES.

1. The line joining the middle points of two parallel chords of a circle passes through the centre.
2. The chords that join the extremities of two equal arcs of a circle towards the same parts are parallel.
3. The sum of the angles subtended at the centre of a circle by two opposite sides of a circumscribed quadrilateral is equal to two right angles.

4. An isosceles triangle has its vertical angle equal to the exterior angle of an equilateral triangle. Prove that the radius of the circumscribing circle is equal to one of the equal sides of the given triangle.

5. Prove that the radius of the circle inscribed in an equilateral triangle is equal to one-third of the altitude of the triangle.

6. The quadrilateral ABCD is inscribed in a circle; AB, DC produced meet in E: prove that the triangles ACE, BDE, and also the triangles ADE, BCE, are mutually equiangular.

7. The quadrilateral ABCD is inscribed in a circle; AB, DC produced meet in E, and BC, AD produced meet in F; the sides AB, BC, CD subtend arcs of 120° , 70° , 80° , respectively: find the number of degrees in the angles AED and AFB.

8. In the circumscribed quadrilateral ABCD, the angles A, B, C are 110° , 95° , 80° , respectively, and the sides AB, BC, CD, DA touch the circumference at the points E, F, G, H, respectively: find the number of degrees in each angle of the quadrilateral EFGH.

9. If two opposite sides of an inscribed quadrilateral are equal, prove that the other two sides are parallel.

PROBLEMS OF CONSTRUCTION.

253. Hitherto, our investigations have been purely *theoretical*, and have been confined to the demonstration of certain properties of figures, assumed to exist, satisfying certain conditions; and our figures have been assumed to be constructed under these conditions, although no methods of constructing them have been given. Indeed, the *precise* construction of the figures was not necessary, as they were required only as *aids* in following the demonstration of *principles*.

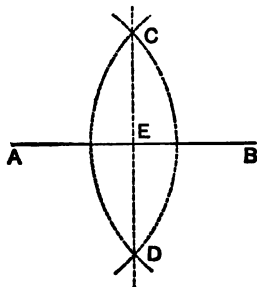
The constructions were only *hypothetical*. We now apply these principles to determine methods by means of which such figures are to be approximately drawn; for, owing to the imperfection of our instruments, the ideal state contemplated in theoretical Geometry cannot be attained by them.

The determination of the method of constructing a given figure with given instruments is called a *problem*; and the *solution* of a problem requires us to *show how* the construction can be affected by the use of the given instruments, and to *prove that* the construction is correct. The solution of a problem depends on the instruments that are used. The more restricted the choice of instruments, the more limited will be the problems which can be solved by their use; and the more difficult will be the solution of many that are thus solved.

It is the recognized convention of Elementary Geometry that the only instruments to be employed are the ruler and compasses, with the use of which the student should become familiar. The ruler is used for drawing and producing straight lines, and the compasses for describing circles and for the transference of distances; the straight line and the circumference being the only lines treated of in this subject. This convention is embodied in the three postulates given in (45).

Problems, though important as applications of geometric truths, form no part of the chain of connected truths embodied in the theorems of Geometry, so that, though they may be studied advantageously in connection with the theorems on which they directly depend, they are not a necessary part of the pure science of Geometry.*

* Elements of Plane Geometry, Association for the improvement of Geometric Teaching, p. 69.

Proposition 21. Problem.**254.** *To bisect a given finite straight line.**Given, the line AB.**Required, to bisect it.*

Cons. With A and B as centres, and equal radii greater than one-half of AB, describe two arcs cutting each other at C and D.

Join CD cutting AB at E.

Then AB is bisected at E.

Proof. Since the two pts. C and D are equally distant from A and B, the st. line CD is $\perp$ to AB at its middle pt.

(67)

Q. E. F.

EXERCISES.

1. If two chords intersect at right angles within a circle, prove that the sum of the opposite intercepted arcs is equal to a semi-circumference.

2. If the angles A, B, C of a circumscribed quadrilateral ABCD are 120° , 80° , and 100° , and the sides AB, BC, CD, DA touch the circumference at the points E, F, G, H, find the number of degrees in each angle of the quadrilateral EFGH.

3. Divide a line into four equal parts.

Proposition 22. Problem.

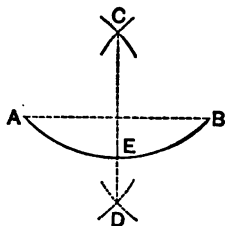
255. *To bisect a given arc or a given angle.*

(1) *Given*, the arc AB.

Required, to bisect it.

Cons. Join AB.

With A and B as centres, and with equal radii, describe arcs intersecting at C and D.



Draw CD cutting the arc AB at E.

Then the arc AB is bisected at E.

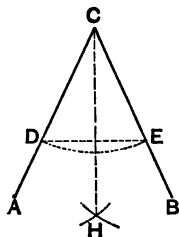
Proof. Since the two pts. C and D are equally distant from A and B, the st. line CD is $\perp$ to the chord AB at its mid. pt. (67), and therefore bisects the arc (202).

(2) *Given*, the $\angle ACB$.

Required, to bisect it.

Cons. With C as a centre, and with any radius, describe an arc cutting CA and CB at D and E.

With D and E as centres, and with a radius $> \frac{1}{2}$ the st. line DE, describe two arcs intersecting at H.



Join CH.

Then CH bisects the $\angle ACB$.

Proof. Join DE. Since the two pts. D and E are equally distant from C and H, the st. line CH is $\perp$ to the chord DE at its mid. pt. (67), and therefore bisects the arc DE and the $\angle DCE$ (202). Q.E.F.

EXERCISES.

1. Divide an arc into four equal parts.
2. Divide an angle into four equal parts.

Proposition 23. Problem.

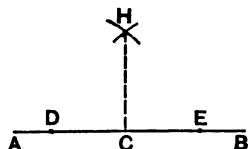
256. *At a given point in a given straight line to draw a perpendicular to that line.*

Given, the pt. C in the line AB.

Required, to draw from C a $\perp$ to AB.

FIRST METHOD.

Cons. With C as a centre, and with any radius, describe arcs of a $\odot$ cutting AB at D and E.



With D and E as centres, and with equal radii greater than DC, describe two arcs cutting each other at H.

Join HC.

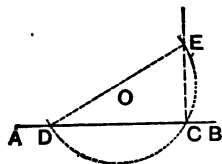
Then HC is the required $\perp$.

Proof. Since the two pts. C and H are equally distant from D and E, the st. line CH is $\perp$ to DE at its mid. pt. C. (67)

SECOND METHOD.

Cons. With any pt. O without AB as a centre, and with the distance OC as a radius, describe a $\odot$ cutting AB at C and D.

Join DO, and produce it to meet the $\odot$ at E. Join EC.

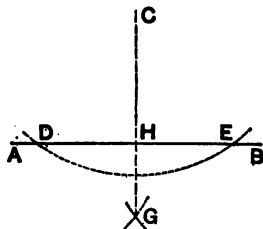


Then EC is the required $\perp$.

Proof. Since the $\angle ACE$ is inscribed in a semicircle, it is a rt. $\angle$; $\therefore$ EC is $\perp$ to AB at C (240). Q.E.F.

Proposition 24. Problem.

257. *From a given point without a given straight line to draw a perpendicular to that line.*



Given, the pt. C and the line AB.

Required, to draw a $\perp$ from C to AB.

Cons. With C as a centre, and with a radius sufficiently great, describe an arc cutting AB at D and E.

With D and E as centres, and with a radius $> \frac{1}{2}$ of DE, describe arcs cutting each other at G.

Join CG, cutting AB at H.

Then CH is the required $\perp$.

Proof. Since the two pts. C and G are equally distant from D and E, the st. line CG is $\perp$ to DE at its mid-pt. (67)

Q. E. F.

EXERCISES.

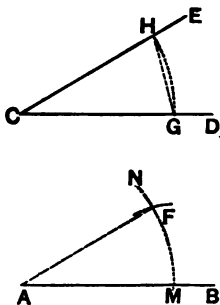
1. In an indefinite straight line AB find a point equally distant from two given points which are not *both* on AB.

When does this problem *not* admit of solution?

2. In a given straight line MN find a point P such that PA, PB, drawn from P to two given points A, B, on opposite sides of MN, may make equal angles with MN.

Proposition 25. Problem.

258. *At a given point on a given straight line to construct an angle equal to a given angle.*



Given, the pt. A, the line AB, and the $\angle C$.

Required, to make at A an $\angle = \angle C$.

Cons. With centre C and any radius, describe an arc cutting CD and CE at G and H. Join GH.

With centre A and the same radius CG, describe an indefinite arc MN.

With centre M and radius = GH, describe an arc cutting the arc MN at F.

Join AF.

Then $\angle A = \angle C$.

Proof. Since chord MN = chord GH, (Cons.)

$\therefore$ arc MN = arc GH, and $\angle A = \angle C$.

In equal $\odot$ s equal chords subtend equal arcs, and equal $\angle$ s at the centre (198).

Q. E. F.

EXERCISES.

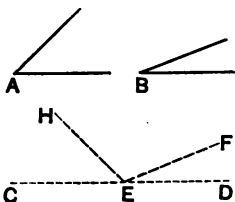
1. Given an angle, to construct its supplement.
2. Construct an angle of 45° .
3. Construct an angle of $22\frac{1}{2}^\circ$.

Proposition 26. Problem.

259. *Two angles of a triangle being given to find the third.*

Given, $\angle A$ and $\angle B$ of a $\triangle$.

Required, to find the third $\angle$ of the $\triangle$.



Cons. Draw the indefinite straight line CD. At any pt. E in this line,

make $\angle DEF = \angle B$,

and $\angle CEH = \angle A$.

(258)

Then $\angle FEH$ is the $\angle$ required.

Proof. Since $\angle CEH + \angle HEF + \angle FED = 2 \text{ rt. } \angle s$, (53)

and $\angle A + \angle B + \text{required } \angle = 2 \text{ rt. } \angle s$, (97)

and $\angle A = \angle CEH$, and $\angle B = \angle FED$, (Cons.)

$\therefore \angle HEF$ is the $\angle$ required.

Q. E. F.

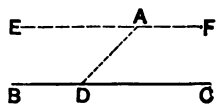
Proposition 27. Problem.

260. *Through a given point to draw a straight line parallel to a given straight line.*

Given, the pt. A and the line BC.

Required, to draw through A a line $\parallel$ to BC.

Cons. In BC take any pt. D, and join DA.



At pt. A make $\angle DAE = \angle ADC$.

(258)

Then EAF is $\parallel$ to BC.

Proof. Since $\angle DAE = \angle ADC$,

(Cons.)

$\therefore EF$ is $\parallel$ to BC,

being alt.-int. $\angle s$. (75).

Q. E. F.

Proposition 28. Problem.

261. *Given two sides and the included angle of a triangle, to construct the triangle.*

Given, two sides a , b , and the included $\angle A$.

Required, to construct the Δ .

Cons. Draw the line $AB = a$.

At A make the $\angle BAD = \angle A$. (258)

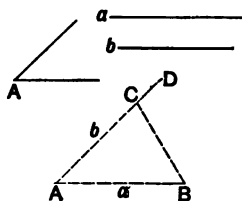
On AD take $AC = b$.

Join CB .

Then ABC is the Δ required.

Proof. To be supplied by the student.

Q. E. F.

**Proposition 29. Problem.**

262. *Given a side and the two adjacent angles of a triangle, to construct the triangle.*

Given, the side a and the adj. $\angle$ s A , B .

Required, to construct the Δ .

Cons. Draw the line $AB = a$.

At A make the $\angle BAD = \angle A$. (258)

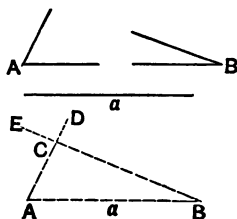
At B make the $\angle ABE = \angle B$. (258)

The lines AD and BE will intersect at some pt. C .

Then ABC is the Δ required.

Proof. To be supplied by the student.

Q. E. F.

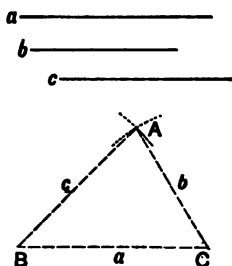


263. SCH. 1. Since the third angle of a triangle can be found when two angles are given (259), therefore, if a side and any two angles of a triangle are given, the triangle may always be constructed. (262)

264. SCH. 2. This problem is possible only when the two given angles are together less than two right angles.

Proposition 30. Problem.

265. *Given the three sides of a triangle to construct the triangle.*



Given, the three sides a , b , c .

Required, to construct the $\triangle$.

Cons. Draw the line $BC = a$.

With C as a centre, and a radius $= b$, describe an arc.

With B as a centre, and a radius $= c$, describe an arc, cutting the former arc at A .

Join BA and CA .

Then ABC is the $\triangle$ required.

Proof. To be supplied by the student.

Q. E. F.

EXERCISES.

1. Show how the construction fails if one side is greater than the sum of the other two.
2. Construct an equilateral triangle on a given base.

Proposition 31. Problem.

266. To construct a triangle having given two sides and the angle opposite one of them.

Given, the sides a , b , and $\angle A$
opp. a .

Required, to construct the $\triangle$.

CASE I. When $a < b$, and $\angle A$ is acute.

Cons. At pt. A on the indefinite st. line AD, make the $\angle DAE = \angle A$. (258)

On AE take $AC = b$.

With C as a centre, and radius $= a$, describe an arc cutting AD at B and B'.

Join CB and CB'.

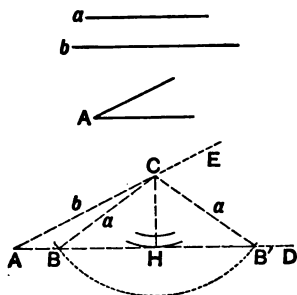
Then, either ABC or AB'C is the $\triangle$ required, since each satisfies the given conditions; and the problem has two solutions.

This is called the *ambiguous case*.

When the side $a =$ the $\perp$ CH, there is but one solution: the rt. $\triangle$ ACH, since the arc described with centre C and radius a touches AD.

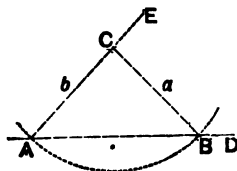
When the side $a < CH$ the problem is impossible, since the arc described with centre C and radius a does not cut or touch AD.

Hence, the arc described with centre C and radius a will meet AD in two pts. B and B' equidistant from H, or in one pt. H, or in no pt. whatever, according as $a >$, $=$, or $<$ CH. Therefore, there may be two $\triangle$ s, one $\triangle$, or none at all.



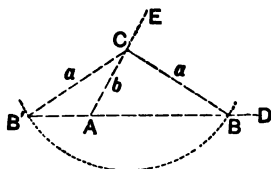
CASE II. When $a = b$, and $\angle A$ is acute.

In this case the arc described with centre C and radius a , cuts AD at the two pts. A and B; and hence there is but one solution: the isosceles $\triangle ABC$.



CASE III. When $a > b$.

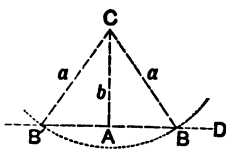
When $\angle A$ is *acute* the arc described with centre C and radius a cuts AD in B and B', the latter pt. at the *left* of A in DA produced. Then ABC is the $\triangle$ required, since it satisfies the given conditions, and there is only one solution, for the $\triangle AB'C$ does not contain the given $\angle A$.



When $\angle A$ is *obtuse*, as $\angle CAB'$, there is only one solution: the $\triangle AB'C$, since the $\triangle ABC$ does not contain the given obtuse $\angle$.

When $\angle A$ is *right* there are two equal rt. $\triangle$ s: ABC and AB'C.

When $\angle A$ is right or obtuse, and $a =$ or $< b$, the problem is impossible; for the side opposite the right or the obtuse $\angle$ is the greatest side of the $\triangle$.



(117)

Q.E.F.

EXERCISES.

1. Construct a right triangle having given the hypotenuse and one side.
2. Construct an isosceles right triangle on a given straight line as hypotenuse.

Proposition 32. Problem.

267. *Given two adjacent sides and the included angle of a parallelogram, to construct the parallelogram.*

Given, the sides a , b , and the $\angle A$.

Required, to construct the $\square$.

Cons. Draw $AB = a$.

At A make the $\angle BAE = \angle A$.

On AE take $AC = b$.

With C as a centre, and a radius $= a$, describe an arc.

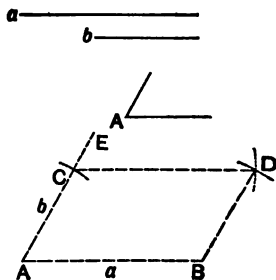
With B as a centre, and a radius $= b$, describe an arc cutting the first at D .

Join CD , BD .

Then $ABCD$ is the required $\square$.

Proof. Since $AB = CD$ and $AC = BD$ (Cons.), the figure is a $\square$ (132), and it is the one required; for two $\square$ s are equal when they have two adj. sides and the included $\angle$ equal, each to each. (135)

Q.E.F.

**EXERCISES.**

1. Construct a square upon a given straight line.
2. Construct a parallelogram, having given two adjacent sides and one diagonal.
3. Construct a rhombus, having given the two diagonals.
4. On a given straight line as hypotenuse, construct a right triangle having one of its acute angles double the other.

Proposition 33. Problem.

268. *To draw a tangent to a given circle from a given point either on or without the circumference.*

Given, the $\odot$ BCD, and the pt. A.

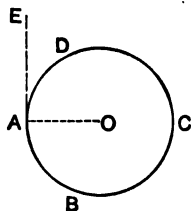
Required, to draw from A a tangent to the $\odot$ BCD.

CASE I. *When the pt. A is in the $\odot$ ce.*

Cons. Draw the radius OA.

At A draw $AE \perp$ to AO. (256)

Then AE is the tangent required. (209)



CASE II. *When the pt. A is without the $\odot$.*

Cons. Join OA; bisect it at E.

With centre E and radius EO, describe a $\odot$ ce, cutting the $\odot$ BCD at B, D.

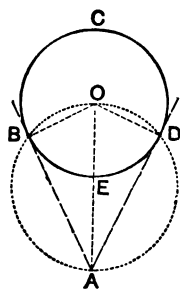
Join AB, AD.

Then either AB or AD is the tangent required.

Proof. Join OB, OD.

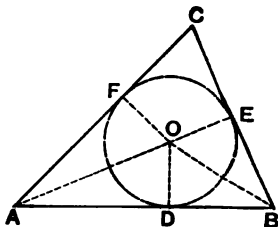
$\angle ABO = \text{rt. } \angle$, and $\angle ADO = \text{rt. } \angle$,
being inscribed in a semicircle (240).

$\therefore$ AB and AD are each tangent to the $\odot$ at B, D,
being $\perp$ to the radius at its extremity (209).



Q. E. F.

SCH. When the pt. A is without the $\odot$, two tangents may always be drawn, and they will be equal; for
 $\text{rt. } \triangle AOB = \text{rt. } \triangle AOD$. (110)

Proposition 34. Problem.**269.** *To inscribe a circle in a given triangle.**Given,* the $\triangle ABC$.*Required,* to inscribe a $\odot$ in $\triangle ABC$.*Cons.* Bisect the $\angle$ s A, B by the lines AO, BO meeting at O.From O draw $OD \perp$ to AB.With centre O and radius OD, describe the $\odot$ DEF.The $\odot$ DEF is the required $\odot$.

Proof. Since the pt. of intersection of the bisectors of the angles of a $\triangle$ is equally distant from the three sides of the $\triangle$,

(167)

 $\therefore$ the $\perp$ s OD, OE, OF are equal.

$\therefore$ a $\odot$ described with centre O and radius OD will be tangent to the three sides of the $\triangle$ at the pts. D, E, F, and be inscribed in it.

(191)

Q. E. F.

270. SCH. The centre of the circle inscribed in a triangle is sometimes called its *in-centre*.

NOTE.—The bisectors of the angles of a triangle are concurrent, the point of intersection being the centre of the circle inscribed in the triangle.

271. DEF. If the sides of a triangle are produced and the exterior angles are bisected, the intersections of the bisectors are the centres of three circles, each of which is tangent to one side of the triangle and the other two sides produced. These three circles are called *escribed circles*.

Proposition 35. Problem.

272. *To draw an escribed circle of a given triangle.*

Given, the $\triangle ABC$.

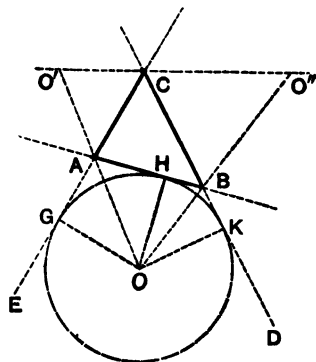
Required, to describe a $\odot$ touching AB, and CA, CB produced.

Cons. Bisect the $\angle$ s BAE, ABD by the lines AO, BO, which intersect at O. (255)

From O draw OG, OH, OK $\perp$ to CE, AB, CD.

With centre O and radius OG, describe the $\odot$ GHK.

The $\odot$ GHK is the required $\odot$.



Proof. Since the pt. O is on the bisector of $\angle$ BAE, it is equally distant from BA and AE. (160)

$$\therefore OG = OH.$$

Similarly, $OH = OK$.

$\therefore$ the $\perp$ s OG, OH, OK are equal.

$\therefore$ a $\odot$ described with centre O and radius OG will touch CE, AB, CD at the pts. G, H, K.

$\therefore$ the $\odot$ GHK is an escribed $\odot$ of the $\triangle ABC$. Q.E.F.

SCH. In the same manner the centres O' , O'' of the other two escribed $\odot$ s may be found.

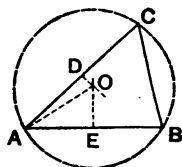
Therefore there are in general four circles tangent to three intersecting straight lines.

Proposition 36. Problem.**273.** *To circumscribe a circle about a given triangle.**Given*, the $\triangle ABC$.*Required*, to circumscribe a $\odot$ about the $\triangle ABC$.*Cons.* Draw DO , $EO \perp$ to AC , AB at their mid. pts. intersecting at O (284).With O as a centre, and a radius $= OA$, describe a $\odot$.This $\odot$ is the one required passing through the vertices A , B , C .*Proof.* Since the pt. O is in the $\perp$ bisectors OD , OE of AC , AB , it is equally distant from the pts. A , B , C .*Every pt. in the $\perp$ bisector of a line is equidistant from the extremities of the line* (66). $\therefore$ a $\odot$ described with centre O and radius OA must pass through the pts. A , B , C , and is $\therefore$ circumscribed about the $\triangle ABC$.

Q. E. F.

274. SCH. This construction is the same as that of describing a circumference through any three given points not in the same straight line, or of finding the centre of a given circle, or of a given arc.**NOTE.**—Since the perpendicular bisector of a chord passes through the centre of the circle (202), therefore,*The perpendicular bisectors of the sides of a triangle are concurrent, the point of intersection being the centre of the circle circumscribed about the triangle.*The centre of the circle circumscribed about a triangle is sometimes called its *circum-centre*.**EXERCISE.**

Prove (1) if the given triangle be acute-angled, the centre of the circumscribed circle falls within it; (2) if it be a right triangle, the centre falls on the hypotenuse; (3) if it be an obtuse-angled triangle, the centre falls without the triangle.



Proposition 37. Problem.

275. *On a given straight line, to describe a segment which shall contain a given angle.*

Given, the st. line AB.

Required, to describe on AB a segment containing $\angle C$.

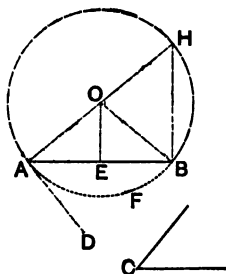
Cons. Make $\angle BAD = \angle C$. (258)

From A draw $AH \perp$ to AD. (256)

Bisect AB in E. (254)

From E draw $EO \perp$ to AB meeting AH at O. (256)

With O as a centre, and OA as a radius, describe the $\odot$ ce AHBF.



The segment AHB is the segment required.

Proof. Join OB and BH. Since O is in the $\perp$ bisector of AB, it is equally distant from the pts. A, B. (66)

$\therefore$ a $\odot$ described with centre O and radius OA must pass through B.

Also, since AD is $\perp$ to AH at its extremity,

$\therefore$ AD is tangent to the $\odot$ at A. (209)

Since AB is a chord through A,

$\therefore \angle BAD$ is measured by $\frac{1}{2}$ arc AFB. (243)

But $\angle AHB$ is measured by $\frac{1}{2}$ arc AFB. (238)

$\therefore$ any $\angle$ in the segment AHB $= \angle C$. (Ax. 1)

Q. E. F.

NOTE.—In the particular case when the given angle C is a right angle, the segment required will be the semicircle described on the given st. line AB.

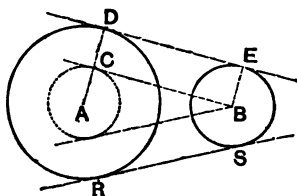
Proposition 38. Problem.

276. To draw a common tangent to two given circles.

Given, the $\odot$ s AR, BS, and
let $AR > BS$.

(1) *Required*, to draw an exterior common tangent to the two $\odot$ s.

Cons. With centre A and radius = the difference of the radii of the two $\odot$ s, describe a $\odot$.



From B draw BC tangent to this $\odot$. (268)

Join AC, and produce it to meet the $\odot$ ce of the given $\odot$ in D.

Draw BE $\parallel$ to AD, and join DE.

DE is a common tangent to the two given $\odot$ s.

Proof. Since $AC = AD - BE$, (Cons.)
 $\therefore CD = BE$.

But CD is $\parallel$ to BE , (Cons.)

$\therefore BCDE$ is a $\square$, (133)

$\therefore DE$ is $=$ and $\parallel$ to CB . (124)

But $\angle ACB = \text{a rt. } \angle$.

A tang. to a $\odot$ at any pt. is $\perp$ to the radius at that pt. (210).

$\therefore BCDE$ is a rectangle. (77)

$\therefore \angle D = \angle E = \text{a rt. } \angle$.

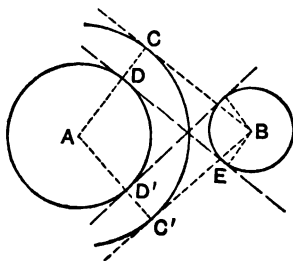
$\therefore DE$ is a tangent to both $\odot$ s. Q. E. F.

Since two tangents can be drawn from B to the $\odot$ AR, therefore two common tangents may always be drawn to the given $\odot$ s. These are called the *direct common tangents*.

When the given $\odot$ s are external to each other and do not intersect, two more common tangents may be drawn, called the *transverse common tangents*.

(2) *Required*, to draw the *transverse* pair of common tangents.

Cons. and proof. With centre A and radius = the sum of the radii of the two $\odot$ s, describe a $\odot$, and complete the construction, and proof, as in (1).



EXERCISES.

THEOREMS.

1. If a straight line cut two concentric circles, the parts of it intercepted between the two circumferences are equal.

2. If one circle touch another internally at P, prove that the straight line joining the extremities of two parallel diameters of the circles, towards the same parts, passes through P.

3. In Ex. 2, if a chord AB of the larger circle touches the smaller one at C, prove that PC bisects the angle APB.

4. If two circles touch externally at P, prove that the straight line joining the extremities of two parallel diameters, towards opposite parts, passes through P.

5. Two circles with centres A and B touch each other externally, and both of them touch another circle with centre O internally : show that the perimeter of the triangle AOB is equal to the diameter of the third circle.

6. In two concentric circles any chord of the outer circle which touches the inner, is bisected at the point of contact.

7. If three circles touch one another externally in P, Q, R, and the chords PQ, PR of two of the circles be produced to meet the third circle again in S, T, prove that ST is a diameter.

8. Points P, Q, R on a circle, whose centre is O, are joined ; OM, ON are drawn perpendicular to PQ, PR respectively : join MN, and show that if the angle OMN

is greater than ONM , then the angle PRQ is greater than PQR .

9. A circle is described on the radius of another circle as diameter, and two chords of the larger circle are drawn, one through the centre of the less at right angles to the common diameter, and the other at right angles to the first through the point where it cuts the less circle. Show that these two chords have their greater segments equal to each other and their less segments equal to each other.

10. O is the centre of a circle, P is any point in its circumference, PN a perpendicular on a fixed diameter: show that the straight line which bisects the angle OPN always passes through one or the other of two fixed points on the circumference.

11. Two tangents are drawn to a circle at the opposite extremities of a diameter, and intercept from a third tangent a portion AB : if C be the centre of the circle show that ACB is a right angle.

12. A straight line touches a circle at A , and from any point P , in the tangent, PB is drawn meeting the circle at B so that PB is equal to PA : prove that PB touches the circle.

13. OC is drawn from the centre O of a circle perpendicular to a chord AB : prove that the tangents at A , B intersect in OC produced.

14. TA , TB are tangents to a circle, whose centre is O ; from a point P on the circumference a tangent is drawn cutting TA , TB , or those produced, in C , D : prove that the angle COD is half the angle AOB .

15. AB is the diameter and C the centre of a semicircle: show that O , the centre of any circle inscribed in the semicircle, is equidistant from C and from the tangent to the semicircle parallel to AB .

16. If from any point without a circle straight lines be drawn touching it, the angle contained by the tangents is double the angle contained by the straight line joining the

points of contact and the diameter drawn through one of them.

17. C is the centre of a given circle, CA a radius, B a point on a radius at right angles to CA ; join AB and produce it to meet the circle again at D , and let the tangent at D meet CB produced at E : show that BDE is an isosceles triangle.

18. Let the diameter BA of a circle be produced to P , so that AP equals the radius; through A draw the tangent AED , and from P draw PEC touching the circle at C and meeting the former tangent at E ; join BC and produce it to meet AED at D : then will the triangle DEC be equilateral.

Let O be the centre of the given $\odot$. Produce OC to a pt. F so that $CF = CO$: compare $\triangle s$ PCO , PCF , etc.

19. APB is a fixed chord passing through P , a point of intersection of two circles AQP , PBR ; and QPR is any other chord of the circles passing through P : show that AQ and RB when produced meet at a constant angle.

20. Two circles whose centres are A and B touch externally at C ; the common tangent at C meets another common tangent DE at F : prove that (1) CF , DF , FE are equal; (2) each of the angles AFB , DCE is a right angle; (3) DE touches the circle described on AB as diameter.

21. The diagonals AC , BD of a quadrilateral $ABCD$ inscribed in a circle intersect at right angles at P : prove that the straight line drawn from P to the middle point of one of the sides of the quadrilateral is perpendicular to the opposite sides.

Bisect AB in E , produce EP to meet CD in F , etc.

22. If a side of a quadrilateral inscribed in a circle be produced, the exterior angle is equal to the interior and opposite angle: and conversely, if the exterior angle of a quadrilateral made by any side and the adjacent side produced be equal to the interior and opposite angle, a circle can be described about the quadrilateral.

23. ABCD is a quadrilateral inscribed in a circle with centre O. If the angles BAD, BOD are together equal to two right angles, prove that the angle BCD is two-thirds of two right angles.

24. If two pairs of opposite sides of a hexagon inscribed in a circle are parallel, the third pair of opposite sides are parallel.

See (242).

25. A circle is described passing through the ends of the base of a given triangle: prove that the straight line joining the points, in which it cuts the sides or sides produced, is parallel to a fixed straight line.

26. In the semicircle ABCDE, the chord BD which is parallel to the diameter AE bisects the radius OC at right angles: prove that the arc BC is double the arc AB.

Join BO, CD; let OC cut BD in F; $BF = FD$, etc.

27. The straight lines which bisect the vertical angles of all triangles on the same base, on the same side of it, and with the same vertical angle, are concurrent.

28. AB, AC are chords of a circle: show that the straight line, which joins the middle points of the arcs AB, AC, cuts off equal portions of the chords.

Let D, E be the mid. pts. of arcs AB, AC; take D, E on opp. sides of diam. through A, etc.

29. BAC, BA'C are two angles in the same segment of a circle; AP, A'P' are drawn making the angles BAP, BA'P' equal to the angles BCA, BCA' respectively: prove that AP is parallel to A'P'.

30. CD is a chord of a circle at right angles to the diameter AB; E is any point in the arc BC; AE cuts CD in F: prove that the angles DFE, ACE are equal.

31. Prove that two of the straight lines which join the ends of two equal chords are parallel, and that the other two are equal.

Let the equal chds. AB, CD be joined towards the same parts by BC, AD; towards opp. parts by AC, BD; $\therefore$ chd. AB = chd. CD, etc.

32. ABCD is a quadrilateral inscribed in a circle. Find the relation between the sides in order that AC may bisect the angle BAD, and BD bisect the angle ABC.

$\angle DAC = \angle CAB$, $\therefore$ arc DC = arc CB, $\therefore$ CB = CD, etc.

33. Two circles touch internally at O, and a line is drawn cutting one of the circles in P, P', the other in Q, Q': show that PQ, P'Q' subtend equal angles at O.

Draw tang. OB, B being towards P, Q, etc.

34. Two circles touch externally at A; the tangent at B to one of them cuts the other in C, D: prove that BC and BD subtend supplementary angles at A.

Draw common tang. AE meeting BC in E, $\angle EAB = \angle EBA$, etc.

35. If the opposite pairs of sides of a quadrilateral in a circle be produced to meet, and the angles so formed be bisected, the bisectors are at right angles to each other.

Let ABCD be a cyclic quadl.; let AD, BC meet in E, and AB, DC in F, etc.

36. If from any point on the circumference of the circle circumscribing a triangle, perpendiculars be drawn to the sides, show that the feet of these perpendiculars are collinear (164).

Let P be any pt. in the $\odot$ ce of the $\odot$ circumscribing the $\triangle ABC$. From P draw PL $\perp$ to BC, PN $\perp$ to AB, etc.

37. In any triangle ABC, if O be the orthocentre (171), and L, M, N the feet of the perpendiculars, the circle described through L, M, N will (1) bisect OA, OB, OC, and (2) will also pass through the middle points D, E, F of the sides of the triangle.

LOCI.

38. Find the locus of a point at a given radial distance from the circumference of a given circle.

See (157).

39. On a given base as hypotenuse right triangles are described: find the locus of their vertices.

See (240).

40. Find the locus of the centre of a circle which passes through two given points.

See (159) and (202).

41. Find the locus of the centre of a circle which touches a given straight line at a given point, or which touches a given circle at a given point.

42. Find the locus of the centre of a circle which touches a given straight line, and has a given radius.

43. Find the locus of the centre of a circle which touches a given circle, and has a given radius.

44. Find the locus of the centre of a circle which touches two given straight lines.

See (162).

45. Prove that the locus of the middle points of equal chords in a given circle is another concentric circle.

46. Prove that the locus of the middle points of all chords drawn through a given point in a given circumference, is a circle passing through the given point and having the radius of the given circle for a diameter.

47. Prove that the locus of the vertex of a triangle having a given base and a given vertical angle, is a circle.

48. A straight line moves so that its ends constantly touch two fixed lines at right angles to each other: prove that the locus of its middle point is a circle: find its centre and radius.

49. A and B are fixed points, and C is any point on a circle; AC is produced to D so that CD is always equal to CB: find the locus of D.

$CD = CB$, $\therefore \angle CDB = \angle CBD$, etc.

50. Through one of the points of intersection of two fixed circles, centres A and B, a chord is drawn meeting the first circle in P and the other in Q. Find the locus of the point of intersection of PA and QB.

Let D be the intersection of $\odot$ s. (1) Let P, Q be each on the arc outside the other $\odot$; let PA, QB meet in R, etc.

51. ABC is a triangle inscribed in a circle, P any point in the circumference of the circle, and Q is a point in PC such that the angle QBC is equal to the angle PBA . Prove that the locus of Q is a circle.

If P is in arc AB , because $\angle QBC = \angle PBA$, and $\angle QCB = \angle PAB$, $\therefore \angle BQC = \angle BPA$, etc.

52. Find the locus of the middle points of all chords of a circle which pass through a given point.

Let P be the given pt., O the cent. of the $\odot$, AB a chd. through P , etc.

53. From any point P on the circumference of a circle circumscribing a triangle ABC , perpendiculars PD , PE are let fall on the sides AB , AC : prove that the locus of the centre of the circle circumscribing the triangle PDE is a circle.

PROBLEMS.

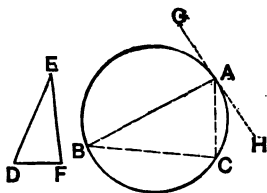
If the construction of a problem does not readily appear to the student, he should present a careful analysis, that is, a course of reasoning by which the solution may be arrived at. In an analysis we generally begin by supposing the problems solved, and constructing the figure accordingly. We then reason out the conditions to be fulfilled, study the relations of the lines, angles, etc., in the figure, draw auxiliary lines parallel or perpendicular, as the case may require, join given points or points assumed in the solution, describe circles when necessary, and endeavor to discover the dependence of the problem upon previously solved problems or theorems. The reverse process, or *synthesis*, then furnishes a construction of the problem. The analysis will furnish exercise for the student's ingenuity in drawing useful auxiliary lines. The synthesis will furnish exercise for his invention in combining the different steps suggested by the analysis in the simplest form.

54. In a given circle to inscribe a triangle equiangular to a given triangle.

Given, the $\odot$ ABC, and the $\triangle$ DEF.

Required, to inscribe in the $\odot$ a $\triangle$ equiangular to DEF.

Analysis. Suppose the problem solved, and that ABC is the required $\triangle$. At A draw the tangent HAG.



Then $\angle GAB = \angle C$, and $\angle HAC = \angle B$, (243)

$\therefore$ the $\angle$ s of the $\triangle$ are the same as the $\angle$ s GAB, BAC, CAH.

Hence, the following construction:

Cons. At any pt. A on the $\odot$ ce, draw the tangent HAG.

At A make $\angle GAB = \angle F$, and $\angle HAC = \angle E$. (258)

Join BC. Then ABC is the inscribed $\triangle$ required. (100)

55. Given two sides of a triangle and the straight line drawn from the extremity of one of them to the middle point of the other, to construct the triangle.

Construct $\triangle$ ABC so that AB = 1st side, BD = $\frac{1}{2}$ the 2nd, etc.

56. Given two sides of a triangle and the straight line drawn from their point of intersection to the middle point of the third side, to construct the triangle.

Construct the $\triangle$ ABE so that AB = 1st side, BE = 2nd side, etc.

57. Given the base, one of the angles at the base, and the sum of the sides of a triangle, to construct the triangle.

58. Given the base, one of the angles at the base, and the difference of the sides of a triangle, to construct the triangle.

59. Through a given point P between two given straight lines AB , AC , draw a straight line which shall be terminated by AB , AC , and bisected in P .

60. If P be outside the lines AB , AC , draw PDE meeting AB , AC in D , E so that PD equals DE .

61. Find points D , E in the sides AB , AC , respectively, of the triangle ABC , so that DE may be parallel to BC and equal to BD .

62. Draw a straight line DE parallel to the base BC of the triangle ABC and meeting AB , AC in D and E , so that DE equals the sum of BD , CE .

63. From a given point without a given straight line draw a straight line making a given angle with this given line.

64. AB , AC are two straight lines, AE is an intermediate straight line. Show how to draw the straight lines which are terminated by AB , AC , and bisected by AE .

Take any pt. E in AE , draw $EF \parallel$ to AC meeting AB in F ; from FB cut off $FG = AF$, etc.

65. Construct a triangle, having given a median and the two angles into which the angle is divided by that median.

66. Construct a parallelogram, having given (1) two diagonals and the angle between them, and (2) one side, one diagonal, and the angle between the diagonals.

67. From a given point O draw three straight lines OA , OB , OC of given lengths so that A , B , C may be collinear, and AB equal to BC .

Construct a $\triangle OAD$ having the sides OA , $AD =$ the two outside lengths, and OD double the middle, etc.

68. In a given straight line find a point whose distance from a given point in the line may be equal to its distance from another given straight line.

Let P be the given pt. in the given line XY , AB the other given line. Draw $PC \perp$ to AB , bisect $\angle XPC$, etc.

69. Construct a triangle, having given the base, the vertical angle, and (1) the sum, or (2) the difference of the sides.

Construct a right triangle having given :

70. The hypotenuse and the sum of the sides.
71. The hypotenuse and the difference of the sides.
72. The hypotenuse and the perpendicular from the right angle on it.
73. The perimeter and an angle.
74. Construct an isosceles triangle, having the vertical angle equal to four times each of the base angles.
75. Construct an isosceles triangle, having one-third of each angle at the base equal to half the vertical angle.

CIRCLES.

76. Through a given point inside a circle which is not the centre, draw a chord which is bisected at that point.
 77. With a given point as centre describe a circle which shall intersect a given circle at the ends of a diameter.
 78. Describe a circle which shall pass through a given point outside or inside a given circle and touch it at a given point.
 79. Describe a circle with a given centre which shall touch a given circle. How many such circles can be drawn?
 80. Through a given point inside a given circle draw two equal chords which shall contain an angle equal to a given angle.
 81. Describe a circle which shall touch a given circle at a given point, and also touch a given straight line.
- Let C be the cent. of $\odot$, P the point. Draw the diam. ABCD $\perp$ to given line XY, meeting the $\odot$ in A; join PA, PB, and produce them to meet XY in E, F, etc.
82. Describe a circle passing through a given point and touching a given straight line at a given point.
 83. With a given point outside a given circle as centre, describe a circle which shall cut the given circle orthogonally (228).
 84. Given two points A, B on a circle, and a fixed straight line through A. Draw through B a straight line

cutting the circle in C, and the fixed line in D, so that AC shall be equal to CD.

Let the fixed line meet the $\odot$ in F; bisect the arc AB in E, draw $EC \perp$ to AF, etc.

85. Describe a circle which shall touch a given straight line at a given point, and pass through another given point not in the line.

86. Construct a triangle, having given the base, the vertical angle, and the median drawn from the vertical angle.

87. ABC is a given straight line: find a point P such that each of the angles APB, BPC may be equal to a given angle.

88. Find the point inside a given triangle at which the sides subtend equal angles.

89. About a given circle to describe a triangle equiangular to a given triangle.

On two sides describe segments of $\odot$ s containing an $\angle = \frac{1}{2}$ of 2 rt. $\angle$ s, etc.

90. If the escribed circles of the triangle ABC (272) touch BC, CA, AB externally in D, E, F respectively, prove that $BD = EA$, $CD = AF$, $CE = BF$.

BOOK III.

RATIO AND PROPORTION. SIMILAR FIGURES.

DEFINITIONS.

277. Four quantities are said to be in *proportion* when the ratio of the first to the second is equal to the ratio of the third to the fourth.

Thus, if $\frac{A}{B} = \frac{C}{D}$,

then, A, B, C, D are said to be in proportion. The proportion is written

$$A : B = C : D,$$

or

$$A : B :: C : D,$$

which is read "A is to B as C is to D."

NOTE.—The first form is preferable, and the one most generally used in the higher mathematics; but the second form is the more usual one in elementary works.

Let a and b denote the numerical measures of A and B (229), and c and d the numerical measures of C and D. Then, since the ratio of two quantities is the same as the ratio of their numerical measures (230),

$$\therefore A : B = a : b,$$

and

$$C : D = c : d.$$

Hence, if four quantities A, B, C, D, are in proportion, their numerical measures a, b, c, d , are in proportion; that is,

$$a : b = c : d.$$

Conversely, if the numerical measures of four quantities A, B, C, D, are in proportion, the quantities themselves are in proportion; that is,

$$A : B = C : D,$$

when A and B are quantities of one kind, and C and D are quantities of one kind, though the latter kind may be different from the former.

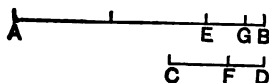
That is, all four quantities *may be* of the same kind, as, for instance, four straight lines, four surfaces, four angles, and so on; but the quantities in *each pair must be* of the same kind.

The magnitudes we meet with in Geometry are more often *incommensurable* (232) than commensurable.

The preceding reasoning does not apply directly to the case in which two quantities are incommensurable, but it may be extended to this case.

278. *To find the greatest common measure of two quantities.*

Let there be two quantities, as, for instance, the two straight lines AB, CD.



Apply the smaller CD to the

greater AB, as many times as possible, suppose twice, with a remainder EB.

Apply the remainder EB to CD as many times as possible, suppose once, with a remainder FD.

Apply the second remainder FD to EB as many times as possible, suppose once, with a remainder GB.

Apply the third remainder GB to FD as many times as possible.

This process will terminate only if a remainder is found which is contained an exact number of times in the preceding one; and if it so terminates the two given lines are commensurable, and the last remainder will be their greatest common measure.

Suppose, for example, that GB is contained exactly twice in FD.

Then GB is contained exactly in CF, and therefore exactly in CD, and therefore exactly in AE, and therefore exactly in AB.

Therefore GB is a common measure (231) of AB and CD.

And, since every common measure of AB and CD must divide AE, it must divide EB or CF, and therefore FD, and therefore also GB. Hence, the common measure cannot be greater than GB. Therefore GB is the greatest common measure of AB and CD.

By regarding GB as the measuring unit, the values of the preceding remainders are easily found, and finally, those of the given lines, from which their numerical ratio is obtained.

$$\text{Thus, } FD = 2GB; \quad EB = FD + GB = 3GB;$$

$$CD = EB + FD = 2FD + GB = 5GB;$$

$$AB = 2CD + EB = 10GB + 3GB = 13GB.$$

Therefore the given lines AB and CD are numerically expressed in terms of the unit GB by the numbers 13 and 5; and their ratio is $\frac{13}{5}$.

279. When the quantities are incommensurable the above process never terminates. However far the operation is continued, we never find a remainder which is contained an exact number of times in the preceding one. But as there is no limit to the number of parts into which AB and CD may be divided, we may obtain a remainder as small as we please, one that is less than any assignable quantity.

Hence, although no *finite* numerator and denominator, however large, can *exactly* express the ratio of two incommensurable quantities, yet by properly increasing both numerator and denominator we may obtain a ratio as nearly

equal as we please to the required ratio, i.e., *we may obtain two numbers whose ratio will express the ratio of two incommensurable quantities to any required degree of accuracy* (232).

NOTE.—We therefore conclude that ratio in Geometry may be treated in the same way as ratio in Arithmetic and Algebra. Indeed, the algebraic treatment is the easiest and the simplest. Euclid's treatment of ratio and proportion is now practically disregarded. While his reasoning is exquisite and rigorous, it is remote from our practical notions on the subject. For these reasons, only simple algebraic proofs of the propositions in proportion will be given in this work. The student will perceive that the propositions do not introduce new ideas, but merely supply proofs, based on the geometric definition of proportion, of results already familiar in the study of Algebra.

280. Let us now consider the numerical proportion (277), which may be written in either of the three forms:

$$a : b = c : d; \quad a : b :: c : d; \quad \frac{a}{b} = \frac{c}{d}.$$

The four terms of the two equal ratios (277) are called the *terms* of the proportion. The first and fourth terms are called the *extremes*, and the second and third, the *means*. Thus, in the above proportion, *a* and *d* are the extremes, and *b* and *c* the means.

The first and third terms are called the *antecedents*, and the second and fourth the *consequents*. Thus, *a* and *c* are the antecedents, and *b* and *d* the consequents.

The fourth term is called a *fourth proportional* to the other three. Thus, in the above proportion, *d* is a fourth proportional to *a*, *b*, and *c*.

In the proportion $a : b = b : c$, *c* is a *third proportional* to *a* and *b*, and *b* is a *mean proportional* between *a* and *c*.

Proposition 1.

281. *If four quantities are in proportion, the product of the extremes is equal to the product of the means.*

Hyp. Let $a : b = c : d$.

To prove $ad = bc$.

Proof. By (277), $\frac{a}{b} = \frac{c}{d}$.

Multiplying by bd , $ad = bc$. Q. E. D.

282. COR. If $a : b = b : c$,
 $\therefore b^2 = ac$. (281)
 $\therefore b = \sqrt{ac}$.

That is, *the mean proportional between two quantities is equal to the square root of their product.*

Proposition 2.

283. *Conversely, if the product of two quantities be equal to the product of two others, two of them may be made the extremes, and the other two the means, of a proportion.*

Hyp. Let $ad = bc$.

To prove $a : b = c : d$.

Proof. Dividing the given equation by bd ,

$$\frac{a}{b} = \frac{c}{d};$$

that is, $a : b = c : d$. Q. E. D.

In a similar manner it may be shown that the proportions

$$a : c = b : d,$$

$$b : a = d : c,$$

$$b : d = a : c,$$

$$c : d = a : b, \text{ etc.,}$$

are all true provided that $ad = bc$.

284. SCH. By the product of the extremes, or of the means of a proportion, is meant the product of the numerical measures of those quantities. Hence, the product of two lines will be often used for brevity, meaning the product of the numbers which represent those lines. (277)

Proposition 3.

285. If four quantities are in proportion, they are in proportion by *inversion*; that is, the second term is to the first as the fourth term is to the third.

Hyp. Let $a : b = c : d.$

To prove $b : a = d : c.$

Proof. $\frac{a}{b} = \frac{c}{d}.$ (277)

$$\therefore 1 \div \frac{a}{b} = 1 \div \frac{c}{d};$$

that is, $\frac{b}{a} = \frac{d}{c}.$

$\therefore b : a = d : c.$ Q. E. D.

Proposition 4.

286. If four quantities are in proportion, they are in proportion by *alternation*; that is, the first term is to the third as the second term is to the fourth.

Hyp. Let $a : b = c : d.$

To prove $a : c = b : d.$

Proof. Since $a : b = c : d,$

$$\therefore ad = bc. \quad (281)$$

$$\therefore a : c = b : d. \quad (283)$$

Q. E. D.

Proposition 5.

287. *If four quantities are in proportion, they are in proportion by composition; that is, the sum of the first and second is to the second as the sum of the third and fourth is to the fourth.*

Hyp. Let $a : b = c : d$.

To prove $a + b : b = c + d : d$.

Proof. $\frac{a}{b} = \frac{c}{d}$. (277)

$$\therefore \frac{a}{b} + 1 = \frac{c}{d} + 1;$$

that is, $\frac{a + b}{b} = \frac{c + d}{d}$.

$\therefore a + b : b = c + d : d$. Q.E.D.

Similarly, $a + b : a = c + d : c$.

Proposition 6.

288. *If four quantities are in proportion, they are in proportion by division; that is, the difference of the first and second is to the second as the difference of the third and fourth is to the fourth.*

Hyp. Let $a : b = c : d$.

To prove $a - b : b = c - d : d$.

Proof. $\frac{a}{b} = \frac{c}{d}$. (277)

$$\therefore \frac{a}{b} - 1 = \frac{c}{d} - 1;$$

that is, $\frac{a - b}{b} = \frac{c - d}{d}$.

$\therefore a - b : b = c - d : d$. Q.E.D.

Similarly, $a - b : a = c - d : c$.

Proposition 7.

289. *If four quantities are in proportion, they are in proportion by composition and division; that is, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference.*

Hyp. Let $a : b = c : d$.

To prove $a + b : a - b = c + d : c - d$.

Proof. By (287), $\frac{a + b}{b} = \frac{c + d}{d}$;

and by (288), $\frac{a - b}{b} = \frac{c - d}{d}$;

by division, $\frac{a + b}{a - b} = \frac{c + d}{c - d}$.

$$\therefore a + b : a - b = c + d : c - d.$$

Q.E.D.

Proposition 8.

290. *The products of the corresponding terms of two or more proportions are proportional.*

Hyp. Let $a : b = c : d$, and $e : f = g : h$.

To prove $ae : bf = cg : dh$.

Proof. $\frac{a}{b} = \frac{c}{d}$, and $\frac{e}{f} = \frac{g}{h}$.

$$\therefore \frac{ae}{bf} = \frac{cg}{dh}.$$

$$\therefore ae : bf = cg : dh.$$

Q.E.D.

291. A greater quantity is said to be a *multiple* of a less, when the greater contains the less an *exact* number of times.

Equimultiples of two quantities are quantities which contain them the same number of times. Thus, ma and mb are equimultiples of a and b .

Proposition 9.

292. *When four quantities are in proportion, if the first and second be multiplied, or divided, by any quantity, as also the third and fourth, the resulting quantities will be in proportion.*

Hyp. Let $a : b = c : d.$

To prove $ma : mb = nc : nd.$

Proof. $\frac{a}{b} = \frac{c}{d}$

Multiply both terms of the first fraction by m , and both terms of the second by n .

Then $\frac{ma}{mb} = \frac{nc}{nd}$

$\therefore ma : mb = nc : nd.$ Q.E.D.

Similarly, $\frac{a}{m} : \frac{b}{m} = \frac{c}{n} : \frac{d}{n}.$

293. SCH. Either m or n may be unity.

In a similar manner it may be shown that *if the first and third terms be multiplied, or divided, by any quantity, and also the second and fourth, the resulting quantities will be in proportion.*

294. COR. Since $\frac{a}{b} = \frac{a}{b},$

$\therefore \frac{ma}{mb} = \frac{a}{b}; \therefore ma : mb = a : b.$

That is, *equimultiples of two quantities are in the same ratio as the quantities themselves.*

Proposition 10.

295. *If four quantities are in proportion, their like powers, or roots, are in proportion.*

Hyp. Let $a : b = c : d$.

To prove $a^n : b^n = c^n : d^n$,

and $a^{\frac{1}{n}} : b^{\frac{1}{n}} = c^{\frac{1}{n}} : d^{\frac{1}{n}}$.

Proof. $\frac{a}{b} = \frac{c}{d}$.

Raising to the n^{th} power,

$$\frac{a^n}{b^n} = \frac{c^n}{d^n}; \therefore a^n : b^n = c^n : d^n.$$

Extracting the n^{th} root,

$$\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}}; \therefore a^{\frac{1}{n}} : b^{\frac{1}{n}} = c^{\frac{1}{n}} : d^{\frac{1}{n}}.$$

Q. E. D.

Proposition 11.

296. *If any number of quantities are in proportion, any antecedent is to its consequent, as the sum of any number of the antecedents is to the sum of the corresponding consequents.*

Hyp. Let $a : b = c : d = e : f = \text{etc.}$

To prove

$$a : b = c : d = \text{etc.} = a + c + e + \text{etc.} : b + d + f + \text{etc.}$$

Proof. $ab = ba$,

$$ad = bc, \text{ and } af = be, \text{ etc., etc.} \quad (281)$$

Adding, $a(b + d + f + \text{etc.}) = b(a + c + e + \text{etc.})$.

$$\therefore a : b = c : d = \text{etc.} = a + c + e + \text{etc.} : b + d + f + \text{etc.} \quad (283)$$

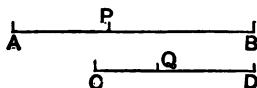
Q. E. D.

PROPORTIONAL LINES.

297. DEF. Two straight lines are said to be *cut proportionally* when the parts of one line are in the same ratio as the corresponding parts of the other line.

Thus, AB and CD are cut proportionally at P and Q if

$$AP : PB = CQ : QD.$$



Proposition 12. Theorem.

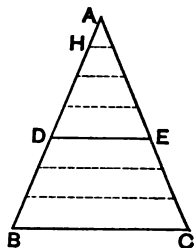
298. *A straight line parallel to one side of a triangle divides the other two sides proportionally.*

Hyp. Let DE be $\parallel$ to BC in the $\triangle ABC$.

To prove $AD : DB = AE : EC$.

CASE I. *When AD and DB are commensurable.*

Proof. Take AH , any common measure of AD and DB , and suppose it to be contained 4 times in AD and 3 times in DB .



$$\text{Then} \quad \frac{AD}{DB} = \frac{4}{3}. \quad (1)$$

Through the several pts. of division of AD , DB draw $\parallel$ s to BC . They will divide AE into 4 equal parts and EC into 3 equal parts.

If $\parallel$ s intercept equal lengths on any transversal, they intercept equal lengths on every transversal (152).

$$\therefore \frac{AE}{EC} = \frac{4}{3}. \quad (2)$$

Therefore, from (1) and (2),

$$\frac{AD}{DB} = \frac{AE}{EC}. \quad (\text{Ax. 1})$$

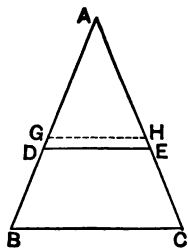
$$\therefore AD : DB = AE : EC.$$

CASE II. *When AD and DB are incommensurable.*

Proof. In this case we know (232) that we may always find a line AG as nearly equal as we please to AD , and such that AG and GB are commensurable.

Draw $GH \parallel$ to BC ; then

$$\frac{AG}{GB} = \frac{AH}{HC}. \quad (\text{Case I.})$$



Now, these two ratios being always equal while the common measure is indefinitely diminished, they will be equal when G moves up to and *as nearly as we please* coincides with D. (233)

$$\therefore \frac{AD}{DB} = \frac{AE}{EC} \quad \text{Q.E.D.}$$

299. COR. 1. By composition (287), we have

$$AD + DB : AD = AE + EC : AE,$$

or $AB : AD = AC : AE.$

Also, $AB : DB = AC : EC, \quad (288)$

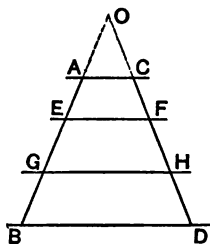
and $AB : AC = DB : EC. \quad (286)$

300. COR. 2. *If two straight lines AB, CD are cut by any number of parallels, AC, EF, GH, BD, the corresponding intercepts are proportional.*

For, let AB and CD meet at O.

Then, by (299),

$$\frac{OE}{OF} = \frac{AE}{CF} = \frac{EG}{FH}.$$



$$\therefore AE : CF = EG : FH.$$

Similarly, $EG : FH = GB : HD.$

EXERCISES.

1. From a point E in the common base AB of two triangles ACB, ADB, straight lines are drawn parallel to AC, AD, meeting BC, BD at F, G : show that FG is parallel to CD.

2. In a triangle ABC the straight line DEF meets the sides BC, CA, AB at the points D, E, F respectively, and it makes equal angles with AB and AC : prove that

$$BD : CD = BF : CE.$$

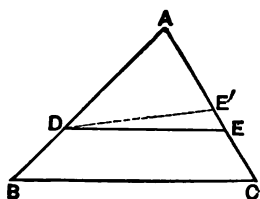
Proposition 13. Theorem.

301. *Conversely, if a straight line divides two sides of a triangle proportionally, it is parallel to the third side.*

Hyp. Let DE cut AB, AC in
the $\triangle ABC$ so that $\frac{AB}{AD} = \frac{AC}{AE}$.

To prove DE $\parallel$ to BC.

Proof. If DE is not $\parallel$ to BC,
draw DE' $\parallel$ to BC.



$$\text{Then,} \quad \frac{AB}{AD} = \frac{AC}{AE'} \quad (299)$$

$$\text{But} \quad \frac{AB}{AD} = \frac{AC}{AE} \quad (\text{Hyp.})$$

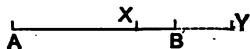
$$\therefore \frac{AC}{AE'} = \frac{AC}{AE} \quad (\text{Ax. 1})$$

$$\therefore AE' = AE,$$

which is impossible unless DE' coincides with DE.

$$\therefore DE \text{ is } \parallel \text{ to } BC. \quad \text{Q. E. D.}$$

302. DEF. When a finite straight line, as AB, is cut at a point X between A and B, it is said to be *divided internally* at X, and the two parts AX and BX are called *segments*. But if the straight line AB is produced, and cut at a point Y beyond AB, it is said to be *divided externally* at Y, and the parts AY and BY are called *segments*. The given line is the *sum* of two *internal segments*, or the *difference* of two *external segments*.



Proposition 14. Theorem.

303. *The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.*

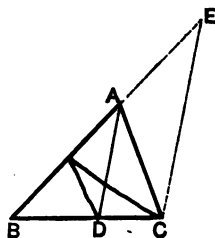
Hyp. Let AD bisect the $\angle A$ of the $\triangle ABC$.

To prove $DB : DC = AB : AC$.

Proof. Draw CE $\parallel$ to DA, meeting BA produced in E.

Since DA is $\parallel$ to CE, (Cons.)

$\therefore \angle BAD = \angle AEC$,
being ext.-int. $\angle$ s of $\parallel$ lines (77),



and $\angle CAD = \angle ACE$,
being alt.-int. $\angle$ s of $\parallel$ lines (72).

But $\angle BAD = \angle CAD$. (Hyp.)

$\therefore \angle AEC = \angle ACE$. (Ax. 1)

$\therefore AE = AC$,
being opp. equal $\angle$ s (114).

Because DA and CE are $\parallel$,

$\therefore DB : DC = AB : AE$. (298)

But $AE = AC$. (Just proved)

$\therefore DB : DC = AB : AC$. Q. E. D.

COR. *Conversely, if AD divides BC into two segments that are proportional to the adjacent sides, it bisects the angle BAC.*

Proposition 15. Theorem.

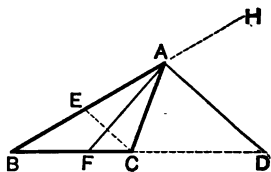
304. *The bisector of an exterior angle of a triangle divides the opposite side externally into segments proportional to the adjacent sides.*

Hyp. Let AD bisect the ext.
 $\angle CAH$ of the $\triangle ABC$.

To prove $DB : DC = AB : AC$.

Proof. Draw CE $\parallel$ to DA
 meeting BA at E.

Since DA is $\parallel$ to CE, (Cons.)



$$\therefore \angle AEC = \angle DAH, \quad (77)$$

$$\text{and} \quad \angle ACE = \angle DAC. \quad (72)$$

$$\text{But} \quad \angle DAH = \angle DAC, \quad (\text{Hyp.})$$

$$\therefore \angle AEC = \angle ACE. \quad (\text{Ax. 1})$$

$$\therefore AE = AC. \quad (144)$$

Because DA and CE are $\parallel$,

$$\therefore DB : DC = AB : AE. \quad (299)$$

$$\text{But} \quad AE = AC. \quad (\text{Just proved})$$

$$\therefore DB : DC = AB : AC. \quad \text{Q. E. D.}$$

305. COR. *If AF be the bisector of the interior angle BAC, we have, from (303) and (304),*

$$BF : FC = BD : CD.$$

306. DEF. When a straight line is divided internally and externally into segments having the same ratio, it is said to be divided *harmonically*. Therefore (305),

The bisectors of an interior and exterior angle at the vertex of a triangle divide the opposite side harmonically.

EXERCISES.

1. If $a : b = e : f$, and $c : d = e : f$, show that $a : b = c : d$.

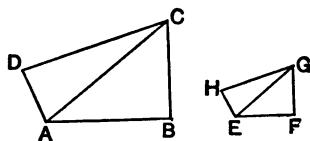
2. If $a : b = b : c$, show that $a : c = a^2 : b^2$.

3. A, B, C, D are four points on a straight line, E is any point outside the line. If the parallelograms AEBF, CEDG be completed, show that FG will be parallel to AD.

SIMILAR FIGURES.

307. DEF. *Similar figures are those which are mutually equiangular (144), and have their homologous sides proportional.*

The homologous sides are those which are adjacent to the equal angles. (145)



Thus, the figures ABCD, EFGH are similar if

$$\angle A = \angle E, \angle B = \angle F, \angle C = \angle G, \text{ etc.,}$$

and

$$\frac{AB}{EF} = \frac{BC}{FG} = \frac{CD}{GH} = \frac{DA}{HE}.$$

The sides AB and EF are homologous, since they are adjacent to the equal angles A and B, and E and F, respectively; also the sides BC and FG are homologous, etc. The diagonals AC, EG are homologous.

308. The constant ratio of any two homologous sides of similar figures is called the *ratio of similitude* of the figures.

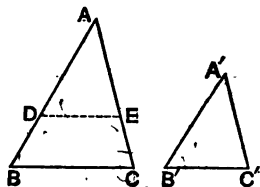
Proposition 16. Theorem.

309. *Triangles which are mutually equiangular are similar.*

Hyp. In the $\triangle$ s ABC , $A'B'C'$,
let $\angle A = \angle A'$, $\angle B = \angle B'$,
 $\angle C = \angle C'$.

To prove $\triangle ABC$ similar to
 $\triangle A'B'C'$.

Proof. Apply the $\triangle A'B'C'$
to the $\triangle ABC$ so that the pt. A'
coincides with A , and $A'B'$ falls on AB . Let B' fall at D .



Then, since $\angle A' = \angle A$, (Hyp.)

$\therefore A'C'$ falls on AC .

Let C' fall at E ; then $B'C'$ falls on DE .

Since $\angle ADE = \angle ABC$, (Hyp.)

$\therefore DE \parallel BC$. (78)

$\therefore AB : AD = AC : AE$, (299)

or $AB : A'B' = AC : A'C'$.

In the same way, by applying the $\triangle A'B'C'$ to the
 $\triangle ABC$ so that the $\angle$ s at B' and B coincide, we may
prove that

$AB : A'B' = BC : B'C'$.

Combining these two proportions, we have

$AB : A'B' = AC : A'C' = BC : B'C'$.

$\therefore$ the two $\triangle$ s are similar. (307)

Q. E. D.

310. COR. 1. *Two triangles are similar when two angles of the one are equal respectively to two angles of the other.* (97)

311. COR. 2. *A triangle is similar to any triangle cut off by a line parallel to one of its sides.*

312. SCH. In similar triangles the homologous sides lie opposite the equal angles.

Proposition 17. Theorem.

313. *Triangles which have their homologous sides proportional are similar.*

Hyp. In the $\triangle s$ ABC , $A'B'C'$,

let
$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}.$$

To prove $\angle A = \angle A'$, $\angle B = \angle B'$, etc.

Proof. On AB take $AD = A'B'$, and draw $DE \parallel$ to BC .

Then, because DE is $\parallel$ to BC ,

$$\therefore \text{the } \triangle s \text{ } ABC, ADE \text{ are similar.} \quad (311)$$

$$\therefore \frac{AB}{AD} = \frac{AC}{AE} = \frac{BC}{DE}. \quad (307)$$

But $A'B' = AD. \quad (\text{Cons.})$

$$\therefore \frac{AB}{AD} = \frac{AC}{A'C'} = \frac{BC}{B'C'}. \quad (\text{Hyp.})$$

Since the first ratio in each of these proportions is the same, the second and third ratios in each are equal respectively; that is,

$$\frac{AC}{AE} = \frac{AC}{A'C'}, \text{ and } \frac{BC}{DE} = \frac{BC}{B'C'}.$$

Since the numerators are the same,

$$\therefore AE = A'C', \text{ and } DE = B'C'.$$

$$\therefore \triangle A'B'C' = \triangle ADE,$$

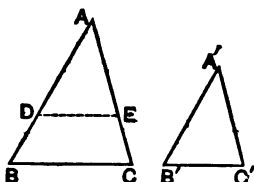
having the three sides equal, each to each (108).

But $\triangle ADE$ is similar to the $\triangle ABC$.

$$\therefore \triangle A'B'C' \text{ is similar to the } \triangle ABC. \quad \text{Q.E.D.}$$

EXERCISE.

If the sides of the triangle in Prop. 14 are $AB = 8$, $BC = 12$, and $AC = 10$, find the lengths of the segments BD and CD .



Proposition 18. Theorem.

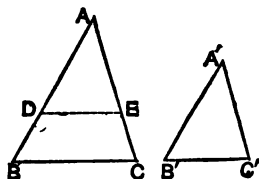
314. *Two triangles which have an angle of the one equal to an angle of the other, and the sides about these angles proportional, are similar.*

Hyp. In the $\triangle s$ ABC , $A'B'C'$, let

$$\angle A = \angle A', \text{ and } \frac{AB}{A'B'} = \frac{AC}{A'C'}.$$

To prove $\triangle ABC$ similar to $\triangle A'B'C'$.

Proof. On AB take $AD = A'B'$, and draw $DE \parallel$ to BC .



Because

DE is $\parallel$ to BC ,

$\therefore$ the $\triangle s$ ABC , ADE are similar. (311)

$\therefore AB : AD = AC : AE$,

homologous sides of similar $\triangle s$ are proportional (307).

But

$A'B' = AD$.

(Cons.)

$\therefore AB : AD = AC : A'C'$.

(Hyp.)

$\therefore A'C' = AE$. (Comparing the two proportions)

$\therefore \triangle A'B'C' = \triangle ADE$,

having two sides and the included $\angle$ equal, each to each (104).

$\therefore \triangle A'B'C'$ is similar to the $\triangle ABC$. Q.E.D.

315. SCH. From the definition (307), it is seen that two conditions are necessary that polygons may be similar: (1) they must be mutually equiangular, and (2) their homologous sides must be proportional. In the case of triangles we learn from Props. 16 and 17 that each of these conditions follows from the other; so that one condition is sufficient to establish the similarity of triangles.

This, however, is not necessarily the case with polygons of more than three sides; for even with quadrilaterals, the angles can be changed without altering the sides, or the proportionality of the sides can be changed without altering the angles.

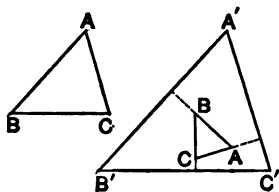
Proposition 19. Theorem.

316. *Two triangles which have their sides parallel or perpendicular, each to each, are similar.*

Hyp. In the $\triangle s$ ABC , $A'B'C'$, let AB , AC , BC be $\parallel$ or $\perp$, respectively, to $A'B'$, $A'C'$, $B'C'$.

To prove $\triangle ABC$ similar to $\triangle A'B'C'$.

Proof. Since the sides are $\parallel$ or $\perp$, each to each, the included $\angle s$ are equal or supplementary (80, 83).



$\therefore$ three hypotheses may be made:

- (1) $A + A' = 2 \text{ rt. } \angle s$, $B + B' = 2 \text{ rt. } \angle s$, $C + C' = 2 \text{ rt. } \angle s$.
- (2) $A = A'$, $B + B' = 2 \text{ rt. } \angle s$, $C + C' = 2 \text{ rt. } \angle s$.
- (3) $A = A'$, $B = B'$, $\therefore C = C'$.

The first and second hypotheses are inadmissible, since the sum of the $\angle s$ of the two $\triangle s$ cannot exceed $4 \text{ rt. } \angle s$ (97). Therefore the third is the only admissible hypothesis.

$\therefore$ the $\triangle s$ ABC , $A'B'C'$ are similar,

being mutually equiangular (809).

Q.E.D.

317. SCH. The homologous sides of the two triangles are either the parallel or the perpendicular sides.

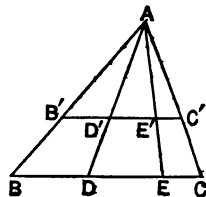
EXERCISE.

If the sides of the triangle in Prop. 15 are $AB = 16$, $BC = 10$, and $AC = 8$, find the lengths of the segments BD and CD .

Proposition 20. Theorem.

318. *If in any triangle a parallel be drawn to the base, all lines from the vertex will divide the base and its parallel proportionally.*

Hyp. Let ABC be a $\triangle$, $B'C' \parallel$ to BC , and AD, AE two lines intersecting $B'C'$ at $D'E'$.



To prove $\frac{BD}{B'D'} = \frac{DE}{D'E'} = \frac{EC}{E'C'}$.

Proof. Since $B'C'$ is $\parallel$ to BC , the $\triangle$ s $AB'D'$, ABD are similar; also the $\triangle$ s $AD'E'$ and ADE ; and the $\triangle$ s $AE'C'$ and AEC . (311)

$$\therefore \frac{AD}{AD'} = \frac{BD}{B'D'}, \text{ and } \frac{AD}{AD'} = \frac{DE}{D'E'},$$

their homologous sides being proportional (307).

$$\therefore \frac{BD}{B'D'} = \frac{DE}{D'E'}. \quad (\text{Ax. 1})$$

In the same way it may be shown that

$$\frac{DE}{D'E'} = \frac{EC}{E'C'}.$$

$$\therefore \frac{BD}{B'D'} = \frac{DE}{D'E'} = \frac{EC}{E'C'}. \quad \text{Q.E.D.}$$

319. COR. If $BD = DE = EC$, we shall have $B'D' = D'E' = E'C'$. Therefore, if the lines from the vertex divide the base into equal parts, they will also divide the parallel into equal parts.

EXERCISES.

1. If $BD = 8$, $B'D' = 6$, $DE = 10$, and $EC = 3$, find the lengths of $D'E'$ and $E'C'$.
2. If $AD = 20$, $AD' = 16$, $AE = 15$, $BD = 7$, and $DE = 5$, find $B'D'$, $D'E'$, and AE' .

Proposition 21. Theorem.

320. *If two polygons are composed of the same number of triangles, similar each to each, and similarly placed, the polygons are similar.*

Hyp. Let the $\triangle$ s ABC, ACD, ADE of the polygon ABCDE be similar respectively to the $\triangle$ s A'B'C', A'C'D', A'D'E' of the polygon A'B'C'D'E', and similarly placed.

To prove the polygons are similar.

Proof. Since the homologous $\angle$ s of similar $\triangle$ s are equal, (307)

$$\therefore \angle B = \angle B'.$$

$$\begin{array}{l} \text{Also,} \quad \angle ACB = \angle A'C'B', \\ \text{and} \quad \angle ACD = \angle A'C'D'. \end{array} \quad (307)$$

$$\text{Adding,} \quad \angle BCD = \angle B'C'D'.$$

In like manner, $\angle CDE = \angle C'D'E'$; etc.

$\therefore$ the polygons are mutually equiangular.

Since similar $\triangle$ s have their homologous sides proportional, (307)

$$\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}, \text{ and } \frac{AC}{A'C'} = \frac{CD}{C'D'},$$

$$\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \text{etc.} \quad (\text{Ax. 1})$$

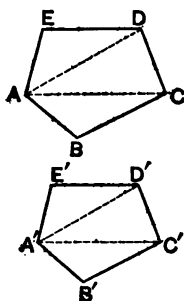
$\therefore$ the homologous sides of the polygons are proportional.

$\therefore$ the polygons are similar,

being mutually equiangular and having their homologous sides

proportional (307).

Q.E.D.



Proposition 22. Theorem.

321. *Conversely, two similar polygons may be divided into the same number of triangles, similar each to each, and similarly placed.*

Hyp. Let $ABCDE$, $A'B'C'D'E'$ be two similar polygons divided into $\triangle$ s by the diagonals AC , AD , $A'C'$, $A'D'$ drawn from the homologous $\angle$ s A and A' .

To prove $\triangle$ s ABC , ACD , ADE similar respectively to $\triangle$ s $A'B'C'$, $A'C'D'$, $A'D'E'$.

Proof. Since the polygons are similar,

$$\therefore \angle B = \angle B', \text{ and } \frac{AB}{A'B'} = \frac{BC}{B'C'}.$$

$$\therefore \triangle ABC \text{ is similar to } \triangle A'B'C'. \quad (314)$$

$$\therefore \angle ACB = \angle A'C'B'. \quad (307)$$

$$\text{But} \quad \angle BCD = \angle B'C'D'. \quad (307)$$

$$\therefore \text{remaining } \angle ACD = \text{remaining } \angle A'C'D'.$$

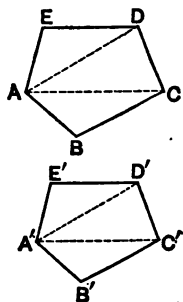
$$\text{Also, } \frac{BC}{B'C'} = \frac{AC}{A'C'}, \text{ and } \frac{BC}{B'C'} = \frac{CD}{C'D'}. \quad (307)$$

$$\therefore AC : A'C' = CD : C'D'. \quad (\text{Ax. 1})$$

$$\therefore \triangle ACD \text{ is similar to } \triangle A'C'D'. \quad (314)$$

In the same way it may be shown that $\triangle$ s ADE , $A'D'E'$ are similar. Q.E.D.

COR. *The homologous diagonals of two similar polygons are proportional to the homologous sides.*



Proposition 23. Theorem.

322. *The perimeters of two similar polygons are to each other as any two homologous sides.*

Hyp. Let $ABCDE$, $A'B'C'D'E'$ be two similar polygons; denote their perimeters by P and P' .

To prove $P : P' = AB : A'B'$.

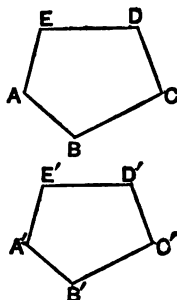
Proof. Since the polygons are similar,

$$\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \text{etc.} \quad (307)$$

$$\therefore \frac{AB + BC + CD + \text{etc.}}{A'B' + B'C' + C'D' + \text{etc.}} = \frac{AB}{A'B'} \quad (296)$$

$$\therefore P : P' = AB : A'B'.$$

Q.E.D.

**EXERCISES.**

1. From the ends of a side of a triangle any two straight lines are drawn to meet the other sides in P , Q ; also from the same ends two lines parallel to the former are drawn to meet the sides produced in P' , Q' : show that PQ is parallel to $P'Q'$.

Let ABC be the Δ ; BP , CQ the lines. $AC : AP = AQ' : AB$, etc.

2. ABC is a triangle, AD any line drawn from A to a point D in BC ; a line is drawn from B bisecting AD in E and cutting AC in F : prove that BF is to BE as $2 CF$ is to AC .

Draw $EG \parallel$ to AC meeting BC in G . $DE = EA$, $\therefore AC = 2 EG$, etc.

3. If in Prop. 20, $AD = 24$, $AD' = 20$, $AE = 16$, $BD = 8$, and $DE = 6$, find $B'D'$, $D'E'$, and AE' .

NUMERICAL RELATIONS BETWEEN THE DIFFERENT PARTS OF A TRIANGLE.

Proposition 24. Theorem.

323. *In a right triangle, if a perpendicular be drawn from the right angle to the hypotenuse:*

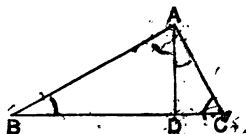
(1) *The two triangles on each side of it are similar to the whole triangle and to each other.*

(2) *The perpendicular is a mean proportional between the segments of the hypotenuse.*

(3) *Each side about the right angle is a mean proportional between the hypotenuse and the adjacent segment.*

Hyp. Let ABC be a rt. $\triangle$, and AD the $\perp$ from the rt. $\angle A$ to the hypotenuse BC .

(1) *To prove* the $\triangle$ s DAB , DAC , and ABC similar.



Proof. In the rt. $\triangle$ s DAB ,

ACB , the acute $\angle B$ is common.

$\therefore$ the $\triangle$ s DAB , ACB are similar. (309)

Also, in the rt. $\triangle$ s DAC , ABC ,

the acute $\angle C$ is common.

$\therefore$ the $\triangle$ s DAC , ABC are similar. (309)

$\therefore$ the $\triangle$ s DAB , DAC are similar to each other, as

they are both similar to ABC . Q.E.D.

(2) *To prove* $BD : AD = AD : CD$.

Proof. Since the $\triangle$ s DAB , DAC are similar,

$\therefore BD : AD = AD : CD$. (307)

(3) *To prove* $BC : AB = AB : BD$.

Proof. Since the $\triangle$ s BAC , BAD are similar,

$$\therefore BC : AB = AB : BD. \quad (307)$$

Also, since the Δ s CAB, CAD are similar,

$$\therefore BC : AC = AC : CD. \quad \text{Q.E.D.}$$

324. COR. 1. If the lines of the figure are expressed in numbers by means of their numerical measures (277), we have from the above three proportions, by means of (281),

$$\overline{AD}^2 = BD \times CD,$$

$$\overline{AB}^2 = BC \times BD,$$

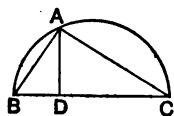
$$\overline{AC}^2 = BC \times CD.$$

By division, the last two equations give

$$\frac{\overline{AB}^2}{\overline{AC}^2} = \frac{BD}{CD}.$$

Hence, *the squares of the sides about the right angle are proportional to the adjacent segments of the hypotenuse.*

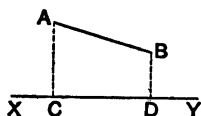
325. COR. 2. Since an angle inscribed in a semicircle is a right angle (240), therefore: (323)



(1) *The perpendicular from any point in the circumference of a circle to a diameter is a mean proportional between the segments of the diameter.*

(2) *The chord from the point to either extremity of the diameter is a mean proportional between the diameter and the adjacent segment.*

326. DEF. *The projection of a point A upon an indefinite straight line XY is the foot C of the perpendicular let fall from the point to the line.*



The projection of a finite straight line AB upon the line XY is the part of XY between the perpendiculars dropped from the ends of the line AB.

Thus, CD is the projection of AB upon the line XY.

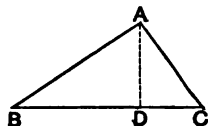
Proposition 25. Theorem.

327. *The square of the number which measures the hypotenuse of a right triangle is equal to the sum of the squares of the numbers which measure the other two sides.*

Hyp. Let ABC be a rt. $\triangle$, with rt. $\angle$ at A .

To prove $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$.

Proof. Draw the $\perp$ AD .



Then $\overline{AB}^2 = BC \times BD$, (324)

and $\overline{AC}^2 = BC \times CD$. (324)

Adding, $\overline{AB}^2 + \overline{AC}^2 = BC(BD + CD) = \overline{BC}^2$. Q. E. D.

328. SCH. 1. By this theorem one of the sides of a right triangle can be found when the other sides are known. If we know the two sides b and c about the rt. $\angle$, the hypotenuse a is given by the formula

$$a^2 = b^2 + c^2. \quad \therefore a = \sqrt{b^2 + c^2}.$$

Thus, if $b = 3$, $c = 4$, we have $a = \sqrt{9 + 16} = \sqrt{25} = 5$.

If the hypotenuse a and one of the sides b of the rt. $\angle$ be known, the other side c is found by the formula

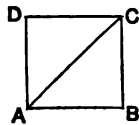
$$c^2 = a^2 - b^2. \quad \therefore c = \sqrt{a^2 - b^2}.$$

Thus, for $a = 5$, and $b = 3$, we have $c = \sqrt{25 - 9} = \sqrt{16} = 4$.

329. SCH. 2. If AC is the diagonal of a square $ABCD$, we have (327)

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2, \text{ or } \overline{AC}^2 = 2 \overline{AB}^2.$$

$$\therefore \frac{\overline{AC}^2}{\overline{AB}^2} = 2, \text{ and } \frac{AC}{AB} = \sqrt{2}.$$



Thus, the diagonal and side of a square are two incommensurable lines, since their ratio, the square root of 2, is an incommensurable number.

NOTE.—To simplify the enunciations of the following theorems, the term *square* of a line will be given to the second power of the number which expresses the measure of the line. The term *product* of two lines will be given to the product of two numbers which express the magnitude of those lines, measured with the same unit.

Proposition 26. Theorem.

330. *In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides diminished by twice the product of one of these sides by the projection of the other side upon it.*

Hyp. Let B be an acute $\angle$ of the $\triangle$ ABC, and BD the projection of AB upon BC.

To prove

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 - 2BC \times BD.$$

Proof. Because ADC is a rt. $\triangle$,

$$\therefore \overline{AC}^2 = \overline{AD}^2 + \overline{DC}^2 \quad (327)$$

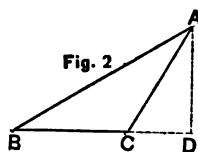
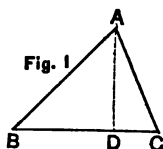
$$= \overline{AD}^2 + (BC - BD)^2 \text{ (fig. 1)}$$

$$\text{or} \quad = \overline{AD}^2 + (BD - BC)^2 \text{ (fig. 2)}$$

$$= \overline{AD}^2 + \overline{BD}^2 + \overline{BC}^2 - 2BC \times BD, \text{ by Algebra*},$$

$$= \overline{AB}^2 + \overline{BC}^2 - 2BC \times BD. \quad (327)$$

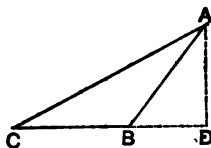
Q.E.D.



* The square of the difference of two numbers is equal to the sum of the squares of the two numbers diminished by twice their product.

Proposition 27. Theorem.

331. *In an obtuse-angled triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides increased by twice the product of one of these sides by the projection of the other side upon it.*



Hyp. Let B be the obtuse $\angle$ of the $\triangle ABC$, and BD the projection of AB upon CB produced.

To prove $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 + 2BC \times BD$.

Proof. $CD = CB + BD$.

$$\therefore \overline{CD}^2 = \overline{CB}^2 + \overline{BD}^2 + 2BC \times BD. \quad (\text{Algebra}^*)$$

Add $\overline{AD}^2$ to both sides.

$$\therefore \overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 + 2BC \times BD. \quad (327)$$

Q.E.D.

332. COR. *From the three preceding theorems, it follows that the square of the side of a triangle is less than, equal to, or greater than, the sum of the squares of the other two sides, according as the angle opposite this side is acute, right, or obtuse.*

NOTE.—By means of these three theorems we may compute the altitudes of a triangle when the three sides are known.

EXERCISES.

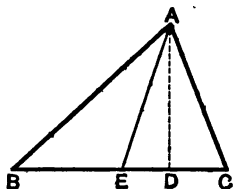
1. If the side BC of an equilateral triangle ABC be produced to D, prove that the square on AD is equal to the squares on DC and CA with the product of DC and CA.

2. If ABC be an isosceles triangle having $AB = AC$, and if D be taken in AC so that $BD = BC$, prove that the square on BC = $AC \times CD$.

* The square of the sum of two numbers is equal to the sum of the squares of the two numbers increased by twice their product.

Proposition 28. Theorem.

333. *The sum of the squares of two sides of a triangle is equal to twice the square of half the third side increased by twice the square of the median upon that side.*



Hyp. Let ABC be a $\triangle$, AE the median bisecting BC .

To prove $\overline{AB}^2 + \overline{AC}^2 = 2\overline{BE}^2 + 2\overline{AE}^2$.

Proof. Draw $AD \perp$ to BC .

Then one of the $\angle$ s AEB , AEC must be obtuse and the other acute: let $\angle AEC$ be acute.

In the $\triangle AEB$, $\overline{AB}^2 = \overline{BE}^2 + \overline{AE}^2 + 2BE \times ED$. (1) (331)

In the $\triangle AEC$, $\overline{AC}^2 = \overline{CE}^2 + \overline{AE}^2 - 2CE \times ED$. (2) (330)

Adding, and observing that $BE = CE$, we get

$$\overline{AB}^2 + \overline{AC}^2 = 2\overline{BE}^2 + 2\overline{AE}^2. \quad \text{Q. E. D.}$$

334. COR. Subtracting (2) from (1) in (333), we have

$$\overline{AB}^2 - \overline{AC}^2 = 2BC \times ED.$$

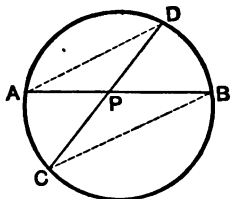
Hence, *the difference of the squares of two sides of a triangle is equal to twice the product of the third side by the projection of the median upon that side.*

Let the student prove the case in which AD falls without the triangle ABC , or coincides with AE .

NOTE.—By means of this theorem we may compute the medians of a triangle when the three sides are known.

Proposition 29. Theorem.

335. *If two chords cut each other in a circle, the product of the segments of the one is equal to the product of the segments of the other.*



Hyp. Let the chords AB, CD cut at P.

To prove $AP \times BP = CP \times DP$.

Proof. Join AD and BC.

In the $\triangle$ s APD, CPB,

$$\angle ADC = \angle ABC, \text{ and } \angle BAD = \angle BCD.$$

$\angle$ s in the same segment of a $\odot$ are equal (239).

$$\therefore \triangle APD \text{ is similar to } \triangle CPB. \quad (309)$$

$$\therefore AP : CP = DP : BP. \quad (312)$$

$$\therefore AP \times BP = CP \times DP. \quad (281)$$

Q. E. D.

336. DEF. When four quantities, such as the sides about two angles, are so related that a side of the first is to a side of the second as the remaining side of the second is to the remaining side of the first, the sides are said to be *reciprocally proportional*. Therefore—

337. COR. 1. *If two chords cut each other in a circle, their segments are reciprocally proportional.*

338. COR. 2. *If through a fixed point within a circle any number of chords are drawn, the products of their segments are all equal.*

EXERCISES.

1. If $AP = 4$, $BP = 5$, and $CD = 12$, find the lengths of CP and DP.

2. If $AB = 20$, and $CD = 24$, find the lengths of CP and DP when CD bisects AB.

Proposition 30. Theorem.

339. *If from a point without a circle a tangent and a secant be drawn, the tangent is a mean proportional between the whole secant and the external segment.*

Hyp. Let PC and PB be a tangent and a secant drawn from the point P to the

⊙ ABC

To prove $PB : PC = PC : PA$.

Proof. Join CA and CB.

In the Δ s PAC, PBC,

$$\angle ACP = \angle ABC,$$

each being measured by $\frac{1}{2}$ arc AC (238), (248).

$\angle P$ is common.

$\therefore \Delta$ s PAC and PBC are similar. (309)

$\therefore PB : PC = PC : PA$. (312)

Q. E. D.

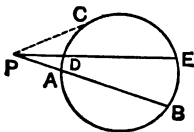
340. COR. 1. $\overline{PC}^2 = PB \times PA$.

Therefore, the square of the tangent is equal to the product of the whole secant and the external segment.

341. COR. 2. Since $PB \times PA = \overline{PC}^2$, (340)

and $PE \times PD = \overline{PC}^2$, (340)

$\therefore PB \times PA = PE \times PD$. (Ax. 1)



Therefore, if from a point without a circle two secants be drawn, the product of one secant and its external segment is equal to the product of the other and its external segment.

342. COR. 3. *If from a point without a circle any number of secants are drawn, the products of the whole secants and their external segments are all equal.*

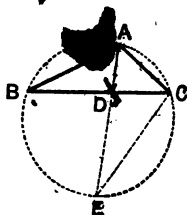
Proposition 31. Theorem.

343. *If any angle of a triangle is bisected by a straight line which cuts the base, the product of the two sides is equal to the product of the segments of the base plus the square of the bisector.*

Hyp. In the $\triangle ABC$ let AD bisect the $\angle BAC$.

To prove $AB \times AC = BD \times DC + \overline{AD}^2$.

Proof. Describe a $\odot$ about the $\triangle ABC$, and produce AD to meet the $\odot$ in E . Join CE .



Then in the $\triangle$ s BAD , EAC ,

$$\angle BAD = \angle EAC. \quad (\text{Hyp.})$$

$$\angle ABD = \angle AEC.$$

$\angle$ s in the same segment of a $\odot$ are equal (239).

$$\therefore \triangle \text{s } BAD \text{ and } EAC \text{ are similar.} \quad (309)$$

$$\therefore AB : AE = AD : AC. \quad (312)$$

$$\therefore AB \times AC = AE \times AD \quad (281)$$

$$= (ED + AD)AD$$

$$= ED \times AD + \overline{AD}^2.$$

$$\text{But} \quad ED \times AD = BD \times DC. \quad (335)$$

$$\therefore AB \times AC = BD \times DC + \overline{AD}^2. \quad \text{Q.E.D.}$$

NOTE.—By means of this theorem we may compute the lengths of the bisectors of the angles of a triangle when the three sides are known.

EXERCISE.

If the vertical angle BAC be externally bisected by a straight line which meets the base in D , show that the product of AB , AC together with the square on AD is equal to the product of the segments of the base.

Proposition 32. Theorem.

344. *The product of two sides of a triangle is equal to the product of the diameter of the circumscribed circle by the perpendicular let fall upon the third side from the vertex of the opposite angle.*

Hyp. In the $\triangle ABC$ let AD be $\perp$ to BC , and AE the diameter of the circumscribed $\odot$.

To prove $AB \times AC = AE \times AD$.

Proof. In the $\triangle s$ ABD , AEC ,

$$\angle B = \angle E,$$

each being measured by $\frac{1}{2}$ arc AC (238),

and $\angle ADB = \angle ACE = \text{a rt. } \angle$. (Cons.) and (240)

$\therefore \triangle s$ ABD , AEC are similar. (309)

$\therefore AB : AE = AD : AC$. (312)

$\therefore AB \times AC = AE \times AD$. (281)

Q. E. D.

NOTE.—By means of this theorem we may compute the radius of the circle circumscribed about a triangle when the three sides are known.

EXERCISE.

The product of the two diagonals of a quadrilateral inscribed in a circle is equal to the sum of the products of its opposite sides.

SUGGESTION.—Make $\angle DAE = \angle BAC$.

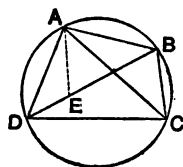
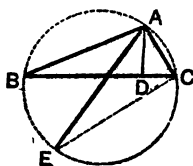
Then in the similar $\triangle s$ DAE , CAB ,

$$AD : AC = DE : BC; \therefore AD \cdot BC = AC \cdot DE.$$

Also in similar $\triangle s$ DAC , EAB ,

$$AB : AC = BE : CD; \therefore AB \cdot CD = AC \cdot BE.$$

Add, $AD \cdot BC + AB \cdot CD = AC \cdot DB$.



PROBLEMS OF CONSTRUCTION.

Proposition 33. Problem.

345. To divide a given straight line into parts proportional to given straight lines.

Given, the line AB, and the lines M, N, P.

Required, to divide AB into parts proportional to M, N, P.

Cons. From A draw the indefinite straight line AH making any $\angle$ with AB.

On AH take

$$AC = M, \quad CD = N, \quad DE = P.$$

Join EB, and through D, C draw DG, CF $\parallel$ to EB.

Then AB is divided at G and F into parts proportional to M, N, P.

Proof. Since CF, DG, EB are $\parallel$, (Cons.)

$$\therefore \frac{AF}{AC} = \frac{FG}{CD} = \frac{GB}{DE}.$$

If two st. lines are cut by any number of $\parallel$ s, the corresponding segments are proportional (300).

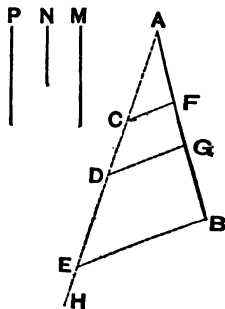
But $AC = M, \quad CD = N, \quad DE = P.$ (Cons.)

$$\therefore \frac{AF}{M} = \frac{FG}{N} = \frac{GB}{P}. \quad \text{Q.E.F.}$$

346. SCH. If it be required to divide a line AB into any number of equal parts, take the same number of equal parts on AH, so that $AC = CD = DE$, and complete the construction as before; then $AF = FG = GB$.

EXERCISES.

1. Divide a straight line into five equal parts.
2. Give the construction for cutting off two-sevenths of a given straight line.

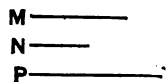


Proposition 34. Problem.

347. To find a fourth proportional to three given straight lines.

Given, the three lines M, N, P.

Required, to find a fourth proportional to M, N, P.

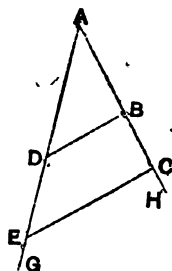


Cons. Draw the two indefinite straight lines AG, AH, making any $\angle$ with each other.

On AH take $AB = M$, $BC = N$; and on AG take $AD = P$.

Join DB, and through C draw $CE \parallel$ to DB.

Then DE is the fourth proportional required.



Proof. Since BD and CE are $\parallel$,

$$\therefore AB : BC = AD : DE.$$

A st. l. $\parallel$ to a side of a $\triangle$ divides the other two sides proportionally (298).

But $AB = M$, $BC = N$, $AD = P$. (Cons.)

$$\therefore M : N = P : DE. \quad \text{Q.E.F.}$$

348. SCH. If the lines N and P are equal, then BC and AD are both laid off equal to N, and DE is the *third* proportional to M and N (280). The proportion in (347) then becomes

$$M : N = N : DE.$$

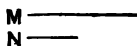
EXERCISES.

1. Construct a fourth proportional to the lines 3, 7, 11.
2. If from D, one of the angles of a parallelogram ABCD, a straight line is drawn meeting AB at E and CB produced at F; show that CF is a fourth proportional to EA, AD, and AB.

Proposition 35. Problem.

349. To find a mean proportional between two given straight lines.

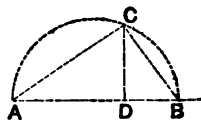
Given, the lines M and N.



Required, to find a mean proportional between M and N.

Cons. On an indefinite straight line take $AD = M$, and $DB = N$.

On AB describe a semicircle, and draw $DC \perp$ to AB.



Then DC is the mean proportional required.

Proof. Since $AD : DC = DC : DB$, (325)

and $AD = M$, $DB = N$, (Cons.)

$\therefore M : DC = DC : N$. Q.E.F.

350. SCH. The mean proportional between two lines is often called the *geometric mean*, while their half sum is called the *arithmetic mean*.

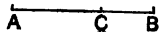
EXERCISE.

From a given point P in a circle a perpendicular PM is drawn to a given chord AB; from A, B perpendiculars AC, BD are drawn to the tangent at P: prove that PM is a mean proportional between AC and BD.

351. DEF. A straight line is said to be divided in *extreme and mean ratio*, when the whole is to the greater segment as the greater segment is to the less.

Thus, the line AB is divided in extreme and mean ratio at C if

$$AB : AC = AC : CB.$$



Proposition 36. Problem.

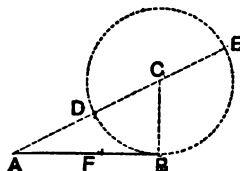
352. To divide a given straight line in extreme and mean ratio.

Given, the st. line AB.

Required, to divide it in extreme and mean ratio.

Cons. At B erect the $\perp$ BC = $\frac{1}{2}$ AB.

With centre C and radius CB describe a $\odot$.



Join AC, cutting the $\odot$ at D, and produce it to meet the $\odot$ again at E.

On AB lay off $AF = AD$.

Then F is the pt. of division required.

Proof. Since AB is a tangent and AE a secant to the $\odot$,
 $\therefore AE : AB = AB : AD$. (1)

The tangent is a mean proportional between the whole secant and the external segment (339).

$\therefore$ by division, $AE - AB : AB = AB - AD : AD$. (288)

But $AB = DE$ and $AD = AF$. (Cons.)

$\therefore AE - AB = AD = AF$, and $AB - AD = FB$.

Substituting these values in the above proportion, we have

$$AF : AB = FB : AF. \quad (285)$$

$$\therefore AB : AF = AF : FB.$$

$\therefore$ AB is divided in extreme and mean ratio at F. Q.E.F.

353. SCH. If BA be produced to the left of A to a point F' so that $AF' = AE$, then F' is another point of division having the same property as F.

Thus we have from proportion (1) in (352), by composition,

$$AE + AB : AE = AB + AD : AB. \quad (287)$$

But $AE + AB = BF'$, and $AB + AD = AF'$. (Cons.)

$$\therefore BF' : AF' = AF' : AB,$$

or

$$AB : AF' = AF' : F'B.$$

$\therefore$ AB is divided in extreme and mean ratio at F'.

AB is said to be divided at F *internally*, and at F' *externally*, in extreme and mean ratio.

NOTE.—This division of the straight line was called by the ancient geometers the *golden section*.

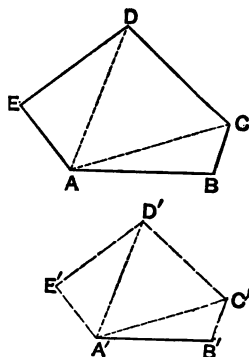
Proposition 37. Problem.

354. *On a given straight line to construct a polygon similar to a given polygon.*

Given, the polygon $ABCDE$ and the straight line $A'B'$.

Required, to construct on $A'B'$ a polygon similar to $ABCDE$.

Cons. Let AB be the side of the given polygon to which $A'B'$ is to be homologous; join AC and AD .



At A' make $\angle B'A'C' = \angle BAC$,

and at B' make $\angle A'B'C' = \angle ABC$.

Also at A' make $\angle C'A'D' = \angle CAD$,

and at C' make $\angle A'C'D' = \angle ACD$.

Also at A' make $\angle D'A'E' = \angle DAE$,

and at D' make $\angle A'D'E' = \angle ADE$.

Then $A'B'C'D'E'$ is the required polygon.

Proof. Since the $\triangle s$ ABC , $A'B'C'$ are mutually equiangular, (Cons.)

$\therefore$ they are similar. (309)

Also, $\triangle s$ ACD , $A'C'D'$ are similar. (309)

Likewise, $\triangle s$ ADE , $A'D'E'$ are similar, (309)

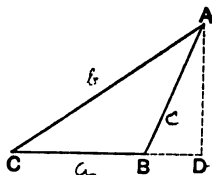
$\therefore$ the polygons $ABCDE$, $A'B'C'D'E'$ are similar,

being composed of the same number of $\triangle s$, similar each to each, and similarly placed (820). Q.E.F.

APPLICATIONS.

1. To find the altitudes of a triangle in terms of its sides.*
See (332).

Let ABC be a $\triangle$; denote by a, b, c , the lengths of the sides opposite the $\angle$ s A, B, C , respectively, and by h the altitude AD .



Of the two $\angle$ s B and C , at least one is acute; suppose it to be the $\angle C$. Then:

$$\text{In the } \triangle ADC, h^2 = b^2 - CD^2. \quad (328)$$

$$\text{In the } \triangle ABC, c^2 = a^2 + b^2 - 2a \times CD. \quad (330)$$

$$\therefore CD = \frac{a^2 + b^2 - c^2}{2a}.$$

$$\begin{aligned} \therefore h^2 &= b^2 - \frac{(a^2 + b^2 - c^2)^2}{4a^2} = \frac{4a^2b^2 - (a^2 + b^2 - c^2)^2}{4a^2} \\ &= \frac{(2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2)}{4a^2} \quad (\text{by Alg. } \dagger) \\ &= \frac{[(a+b)^2 - c^2][c^2 - (a-b)^2]}{4a^2} \\ &= \frac{(a+b+c)(a+b-c)(c+a-b)(c-a+b)}{4a^2}. \quad (1) \end{aligned}$$

$$\text{Put } a + b + c = 2s.$$

$$\text{Then } a + b - c = 2(s - c).$$

$$c + a - b = 2(s - b).$$

$$c - a + b = 2(s - a).$$

Substituting in (1) and extracting the square root, we have

$$h = \frac{2}{a} \sqrt{s(s-a)(s-b)(s-c)}.$$

2. To find the medians of a triangle in terms of its sides.
See (333).

* See *Traité de Géométrie*, par Rouché et Comberousse, p. 139.

† The difference of the squares of two numbers equals the product of the sum and difference of the two numbers.

Denote by m the median on the side BC.

$$\text{Then, } b^2 + c^2 = 2m^2 + 2\left(\frac{a}{2}\right)^2. \quad (333)$$

$$\therefore m = \frac{1}{2} \sqrt{2(b^2 + c^2) - a^2}.$$

3. To find the bisectors of a triangle in terms of its sides. See (343).

Denote the bisector AD, figure of (343), by d , and the sides as in Exs. 1 and 2.

$$\text{By (343), } bc = BD \times DC + d^2.$$

$$\therefore d^2 = bc - BD \times DC. \quad (1)$$

$$\text{By (303), } \frac{BD}{c} = \frac{DC}{b} = \frac{BD + DC}{b + c} = \frac{a}{b + c}. \quad (296)$$

$$\therefore BD = \frac{ac}{b + c}, \text{ and } DC = \frac{ab}{b + c}.$$

Substituting these values in (1) and reducing, and denoting by s the semi-perimeter of the Δ , we have

$$d = \frac{2}{b + c} \sqrt{bcs(s - a)}.$$

4. To find the radius of the circumscribed circle in terms of the sides of the triangle. See (344).

Denote the radius by R , figure of (344), and the sides as before.

$$\text{By (344), } bc = 2R \times AD.$$

Substituting for AD its value in Ex. 1, and solving for R , we have

$$R = \frac{abc}{4 \sqrt{s(s - a)(s - b)(s - c)}}.$$

5. Find the altitudes of a triangle whose sides are 13, 9, and 6. *Ans.* 3.641, 5.259, 7.886.

6. Find the medians of the above triangle.

Ans. 4.031, 9.069, 10.770.

7. Find (1) the bisectors of the angles of the above triangle, and (2) the radius of the circumscribed circle.

Ans. (1) 3.833, 7.778, 10.407; (2) 7.416.

EXERCISES.

THEOREMS.

1. If two circles touch each other, either internally or externally, any two straight lines drawn from the point of contact will be cut proportionally by the circumferences.

2. If two circles intersect in P, and the tangents at P to the two circles meet the circles again in Q and R; prove that PQ:PR in the same ratio as the radii of the circles.

3. ABC, DEF are two isosceles triangles, BC, EF being the bases. If $AB:BC = DE:EF$, show that the triangles are similar.

4. P is a point in a diagonal of a parallelogram. EPF, GPH join points in the opposite sides of the parallelogram. EH, GF cut the diagonal in which P is. Show that EH is parallel to GF.

5. The parallel straight lines AB, CD are joined by AD, BC which intersect in O; in AB, CD points E, F are taken such that $AE:EB = DF:FC$: prove that E, O, F are collinear. (164).

6. The side BC of a triangle ABC is bisected at D, and the angles ADB, ADC are bisected by the straight lines DE, DF, meeting AB, AC at E, F respectively: show that EF is parallel to BC.

Apply (303).

7. AB is a diameter of a circle, CD is a chord at right angles to it, and E is any point in CD; AE and BE are drawn and produced to cut the circle at F and G: show that the quadrilateral CFDG has any two of its sides in the same ratio as the remaining two.

Apply (303).

8. In the circumference of the circle of which AB is the diameter, take any point P, and draw PC, PD on opposite sides of AP, and equally inclined to it, meeting AB at C and D: show that $AC:BC = AD:BD$.

Apply (303) and (304).

9. From the same point A straight lines are drawn making the angles BAC, CAD, DAE each equal to half a right angle, and they are cut by a straight line BCDE, which

makes BAE an isosceles triangle: show that BC or DE is a mean proportional between BE and CD.

Apply (303) and (304).

10. ABCD is a quadrilateral having two of its sides AB, CD parallel; AF, CG are drawn parallel to each other meeting BC, AD, respectively, in F, G. Prove that BG is parallel to DF.

11. If D be the middle point of the side BC of a triangle ABC, and if any straight line be drawn through C meeting AD in E and AB in F; show that the ratio of AE to ED will be twice the ratio of AF to FB.

12. If O be the centre of the inscribed circle of the triangle ABC, and AO meet BC in D; prove that AO is to OD as the sum of AB and AC is to BC.

Apply (303).

13. ABCD is a quadrilateral inscribed in a circle; AD, BC meet in P, and the angle P is bisected by a straight line cutting AB, CD in E, F. Show that $AB : BE = CD : DF$.

Apply (309).

14. ABC is a straight line, and ABD, BCE triangles on the same side of it, such that the angles ABD, ECB are equal, and $AB : BC = BE : BD$: if AE, CD intersect in F, prove that AFC is an isosceles triangle.

See (314).

15. Two circles touch in C, a point D is taken outside them such that the radii AC, BC subtend equal angles at D, and DE, DF are tangents to the circles: if EF cut DG in G, prove that $DE : DF = EG : GF$.

16. C is the centre of a circle, and A any point within it; CA is produced through A to a point B such that the radius is a mean proportional between CA and CB: show that if P be any point on the circumference, the angles CPA and CPB are equal.

See (314).

17. AB, AC are the equal sides of an isosceles triangle; the straight line bisecting AB at right angles meets BC in D: prove that the square on AB = $BC \times BD$.

18. The square on the base of an isosceles triangle is equal to twice the product of either side by the part of that side intercepted between the perpendicular let fall on the side from the opposite angle and the end of the base.

Apply (330).

19. ABC is a triangle having the sides AB, AC equal; if AB is produced to D so that BD is equal to AB , show that the square on CD is equal to the square on AB , together with twice the square on BC .

20. The sum of the squares on the four sides of a parallelogram is equal to the sum of the squares on the diagonals.

Apply (333).

21. The base of a triangle is given and is bisected by the centre of a given circle : if the vertex be at any point of the circumference, show that the sum of the squares on the two sides of the triangle is constant.

22. In any equilateral the sum of the squares on the diagonals is equal to the sum of the squares on the straight lines joining the middle points of opposite sides.

Apply (155) and then Ex. 20.

23. The sum of the squares on the four sides of a quadrilateral is equal to the sum of the squares on the diagonals, increased by four times the square on the line joining the middle points of the diagonals.

Apply (333).

24. AB is a diameter of a circle, the chords AC, BD intersect in a point E inside the circle : prove that the square on AB is equal to the sum of the products of AC, AE and BD, BE .

Apply (335) and (331).

25. AD, BE, CF , the perpendiculars from the vertices on the opposite sides of a triangle ABC , intersect in O : prove that the products of AO, OD and BO, OE and CO, OF are equal to each other.

Apply (335).

26. With a point O inside a circle ABC as centre a circle is described cutting the former circle in D, E ; a chord AOB is drawn to the first circle : if the product AO, OB is

equal to the square on the radius of the second circle, prove that D, O, E are collinear.

Apply (335).

27. If two circles intersect, prove that tangents drawn to them from any point in their common chord produced are equal to each other.

28. AEB, PEDQ are respectively a diameter and a chord of a circle at right angles to each other; ADC, BC are other chords. Prove that the product of AB, AE is equal to the product of AC, AD. (341).

29. If AB be a diameter of a circle, and APQ a straight line cutting the circle again at P and a fixed straight line perpendicular to AB at Q; the product of AP, AQ is constant. (341).

30. A, B, D are three points in order on a circle; C is the middle point of the arc AB; if AC, AD, BD, CD be joined, prove that the square on CD is equal to the square on AC with the product of AD, BD.

31. A circle is described round an equilateral triangle, and from any point in the circumference straight lines are drawn to the angular points of the triangle: show that one of these straight lines is equal to the sum of the other two.

32. From the extremities B, C of the base of an isosceles triangle ABC, straight lines are drawn at right angles to AB, AC respectively, and intersecting at D: show that $BC \times AD = 2 AD \times DB$.

PROBLEMS.

33. AD bisects the angle BAC. From any point in AB draw a straight line to AC which shall be divided by AD in a given ratio, i.e., into parts which have the same ratio which two given straight lines have to each other.

34. Through a given point between two given straight lines draw a straight line which shall be terminated by the given lines and divided by the point in a given ratio.

35. Upon a given portion AC of the diameter AB of a semicircle another semicircle is described. Draw a line

through A so that the part intercepted between the semi-circles may be of given length.

36. ABC is a segment of a circle; the chord AC is divided into two parts at D. Divide the arc ABC into two parts so that the chords of the parts shall have the same ratio which AD, DC have to each other.

Complete the $\odot$ ABCE; bisect arc AC opp. B in E; produce ED to B, etc.

37. Divide the straight line AB into two parts at C so that the square on AC may be equal to twice the square on BC.

Make $\angle BAD = \frac{1}{2}$ rt. $\angle$, draw $BD \perp$ to AB, make $\angle ADC = \angle DAB$, etc.

38. From a given point as centre describe a circle cutting a given straight line in two points, so that the product of their distances from a fixed point in the straight line may be equal to a given square.

39. ABC is a given obtuse-angled triangle: find a point D in the side BC opposite the obtuse angle, so that $BD \times DC = \overline{AD}^2$.

Let O be the cent. of the $\odot$ about $\triangle ABC$. Join AO, on it describe $\frac{1}{2}$ $\odot$ ADO, cutting BC in D; join AD and produce it to E, etc.

40. Describe a circle which shall pass through two given points and touch a given straight line. Two solutions. (339).

Let A, B be the pts., XY the line; produce AB to meet XY in C, etc.

41. Describe a circle to pass through a given point and touch two given straight lines which are not parallel to each other. Two solutions.

Let given pt. P be between the given lines AB, AC; bisect $\angle BAC$, etc.

42. From a given point on the circumference of a given circle draw two chords so as to be in a given ratio and to contain a given angle.

Let P be the given pt.; draw diameter PA; make $\angle APB =$ given $\angle$, etc.

43. From any point A on one of two intersecting straight lines OA, OB draw two straight lines AB, AC cutting OB in B and C, such that the triangles AOB, OAC may be similar, and the sides AB, AC have a given ratio to each other.

44. To describe a circle which shall pass through two given points and touch a given circle.

BOOK IV.

AREAS OF POLYGONS.

MEASUREMENT OF AREAS.

355. DEF. The *area* of a surface is its *numerical measure* (229), referred to the *unit of surface*.

To form a unit of surface, we take any unit of length, as an inch, a foot, etc., and construct a square on it. This square is the corresponding unit of surface, or the *superficial unit*.

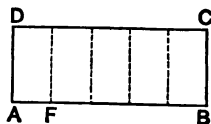
Two surfaces are *equivalent* when they have equal areas and cannot coincide; they are *equal* when they can coincide.

Proposition 1. Theorem.

356. *If two rectangles have equal altitudes, their areas are to each other as their bases.*

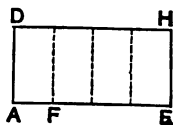
Hyp. Let ABCD, AEHD be two rectangles, having equal altitudes.

To prove
$$\frac{ABCD}{AEHD} = \frac{AB}{AE}.$$



CASE I. When *AB* and *AE* are commensurable.

Proof. Take *AF*, any common measure of *AB* and *AE*, and suppose it to be contained 5 times in *AB* and 4 times in *AE*.



Then
$$\frac{AB}{AE} = \frac{5}{4}.$$

Draw $\perp$ s to *AB* and *AE* at the several points of division. The rectangle *ABCD* will be divided into 5 rectangles, and *AEHD* into 4 rectangles.

These rectangles are all equal. (136)

$$\therefore \frac{ABCD}{AEHD} = \frac{5}{4}.$$

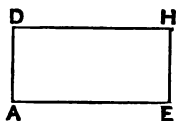
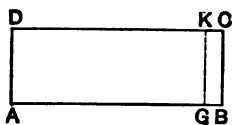
$$\therefore \frac{ABCD}{AEHD} = \frac{AB}{AE}. \quad (\text{Ax. 1})$$

CASE II. When AB and AE are incommensurable.

Proof. In this case we know (232) that we may always find a line AG as nearly equal as we please to AB , and such that AG and AE are commensurable.

Draw $GK \parallel$ to BC .

Then, since AG and AE are commensurable,



$$\therefore \frac{AGKD}{AEHD} = \frac{AG}{AE}. \quad (\text{Case I})$$

Now, these two ratios being always equal while the common measure is indefinitely diminished, they will be equal when G moves up to and as nearly as we please coincides with B . (233)

$$\therefore \frac{ABCD}{AEHD} = \frac{AB}{AE}. \quad \text{Q. E. D.}$$

357. COR. Since the altitudes of rectangles may be considered as their bases, and their bases as their altitudes, therefore :

The areas of rectangles having equal bases are to each other as their altitudes.

NOTE.—In these propositions the words “rectangle,” “triangle,” etc., are often used to denote the “area of the rectangle,” the “area of the triangle,” etc.

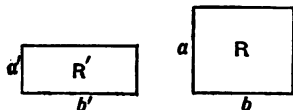
EXERCISES.

The area of a rectangle is 1728 square feet, and its base is 2 yards : what is the area of a second rectangle whose base is 5 yards and whose altitude is the same as that of the first ?

Proposition 2. Theorem.

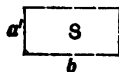
358. *The areas of any two rectangles are to each other as the products of their bases by their altitudes.*

Hyp. Let R and R' be two rectangles, b and b' their bases, a and a' their altitudes,



To prove $\frac{R}{R'} = \frac{a \times b}{a' \times b'}.$

Proof. Construct the rectangle S having the same base b as that of R , and the same altitude a' as that of R' .



Then $\frac{R}{S} = \frac{a}{a'},$

rectangles of equal bases are to each other as their altitudes (357);

and $\frac{S}{R'} = \frac{b}{b'},$

rectangles of equal altitudes are to each other as their bases (356).

Multiplying these ratios, and canceling the common factor, we have

$$\frac{R}{R'} = \frac{a \times b}{a' \times b'}. \quad \text{Q. E. D.}$$

359. SCH. By the product of two lines is meant the product of the numbers which represent those lines when they are measured by a common linear unit. (277)

EXERCISE.

Find the ratio of two rectangles, the base of the first being 3 yards and its altitude 13 feet, the base of the second being 8 feet and its altitude 39 inches.

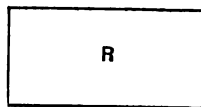
Reduce to the same unit, and compare.

Ans. $4\frac{1}{2}$.

Proposition 3. Theorem.

360. *The area of a rectangle is equal to the product of its base and altitude.*

Hyp. Let R be the rectangle, b the base, and a the altitude expressed in numbers of the same linear unit; and let S be the square whose side is the linear unit.



To prove area of $R = a \times b$.

$$\text{Proof.} \quad \frac{R}{S} = \frac{a \times b}{1 \times 1} = a \times b. \quad (358)$$

But since S is the unit of surface,

$$\therefore R \div S = \text{the area of } R. \quad (355)$$

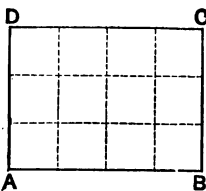
$$\therefore \text{area of } R = a \times b.$$

Q.E.D.

361. SCH. 1. The statement of this Proposition is an abbreviation of the following:

The number of units of area in a rectangular figure is equal to the product of the number of linear units in its base by the number of linear units in its altitude.

When the base and altitude can be expressed *exactly* in terms of some common unit, this proposition is rendered evident by dividing the figure into squares, each equal to the unit of measure. Thus, if AB contain 4 linear units and AD 3, and if through the points of division parallels are drawn, it is seen that the rectangle is divided into *three rows*, each having *four squares*, or into *four columns*, each having *three squares*. Hence, the whole rectangle contains 3×4 , or 12 squares, each equal to the unit of measure.



Similarly, if AB and AD contain m and n units of length respectively, the rectangle will contain mn units of area.

362. SCH. 2. *The area of a square is equal to the second power of its side, being the product of two equal sides.*

In Geometry, the expression "rectangle of two lines" is frequently used for the "product of two lines," meaning the product of their numerical measures.

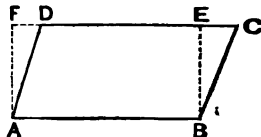
Proposition 4. Theorem.

363. *The area of a parallelogram is equal to the product of its base and altitude.*

Hyp. Let $ABCD$ be a $\square$, AB its base, and BE its altitude.

To prove $\square ABCD = AB \times BE$.

Proof. Draw $AF \perp$ to AB meeting CD produced to F .



Then $ABEF$ is a rectangle having the same base and altitude as the $\square ABCD$.

Then $AD = BC$, and $AF = BE$,
being opp. sides of a $\square$ (129).

$$\therefore \triangle ADF = \triangle BEC,$$

having the hypotenuse and a side respectively equal in each (110).

Take the $\triangle ADF$ from the trapezoid $ABCF$, and there is left the $\square ABCD$.

Take the $\triangle BEC$ from the trapezoid $ABCF$, and there is left the rect. $ABEF$.

$$\therefore \square ABCD = \text{rect. } ABEF. \quad (\text{Ax. 3})$$

$$\text{But area of rect. } ABEF = AB \times BE. \quad (360)$$

$$\therefore \text{area of } \square ABCD = AB \times BE. \quad (\text{Ax. 1})$$

Q. E. D.

364. COR. 1. *Parallelograms having equal bases and equal altitudes are equivalent, because they are all equivalent to the same rectangle.*

365. COR. 2. *Any two parallelograms are to each other as the products of their bases by their altitudes; therefore parallelograms having equal bases are to each other as their altitudes, and parallelograms of equal altitudes are to each other as their bases.*

Proposition 5. Theorem.

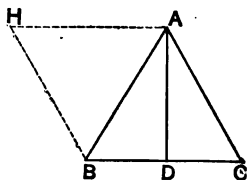
366. *The area of a triangle is equal to half the product of its base and altitude.*

Hyp. Let ABC be a $\triangle$, BC its base, and AD its altitude.

To prove $\triangle ABC = \frac{1}{2} BC \times AD$.

Proof. From B draw $BH \parallel$ to CA , and from A draw $AH \parallel$ to CB meeting BH in H .

$BCAH$ is a $\square$ having the base BC and the altitude AD . (124)



$$\therefore \triangle BAC = \triangle BAH.$$

A diagonal of a $\square$ divides it into two equal $\triangle$ s (130).

But $\square BCAH = BC \times AD$.

The area of a $\square$ is equal to the product of its base and altitude (363).

$$\therefore \triangle ABC = \frac{1}{2} \square BCAH = \frac{1}{2} BC \times AD. \text{ Q.E.D.}$$

367. COR. 1. *Triangles having equal bases and equal altitudes are equivalent; and therefore triangles on the same base, and having their vertices in the same straight line parallel to the base, are equivalent.*

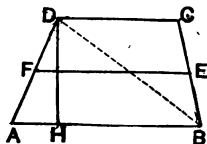
368. COR. 2. *If a triangle and a parallelogram have the same base and are between the same parallels, the area of the triangle is half that of the parallelogram.*

369. COR. 3. *Any two triangles are to each other as the products of their bases by their altitudes; therefore triangles having equal bases are to each other as their altitudes, and triangles of equal altitudes are to each other as their bases.*

370. COR. 4. *Of two parallelograms or two triangles on equal bases, that is the greater which has the greater altitude; and of two parallelograms or triangles of equal altitudes, that is the greater which has the greater base.*

Proposition 6. Theorem.

371. *The area of a trapezoid is equal to the product of the half sum of its parallel sides by its altitude.*



Hyp. Let ABCD be a trapezoid, AB and CD the || sides, and DH the altitude.

To prove area ABCD = $\frac{1}{2} (AB + CD) DH$.

Proof. Draw the diagonal BD, dividing the trapezoid into two $\triangle$ s ABD and DCB, having the common altitude DH.

Then area $\triangle ABD = \frac{1}{2} AB \times DH$,

and area $\triangle DCB = \frac{1}{2} CD \times DH$.

The area of a $\triangle$ equals $\frac{1}{2}$ the product of its base and alt. (366).

$\therefore$ area ABCD = $\frac{1}{2} (AB + CD) DH$,

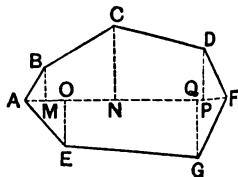
since the two $\triangle$ s make up the area of the trapezoid.

Q.E.D.

372. COR. Since the median EF = $\frac{1}{2} (AB + CD)$ (156), therefore, *the area of a trapezoid is equal to the product of the median joining the middle points of the non-parallel sides by the altitude.*

$\therefore$ area ABCD = FE $\times$ DH.

373. SCH. The area of any polygon may be found by dividing it into triangles, and finding the areas of the several triangles. But in practice the method usually employed is to draw the longest diagonal AF of the polygon, and upon AF let fall the perpendiculars BM, CN, DP, EO, GQ, thus decomposing the polygon into right triangles and trapezoids. By measuring the lengths of the perpendiculars, and the distances between their feet upon AF, the areas of these figures are readily found, and their sum will be the area of the polygon.



COMPARISON OF AREAS.

Proposition 7. Theorem.

374. *The square described on the hypotenuse of a right triangle is equivalent to the sum of the squares described on the other two sides.*

Hyp. Let ABC be a rt. $\triangle$, rt. angled at A , and BE , AK , AF squares on BC , AC , AB .

To prove

sq. on BC = sq. on AB + sq. on AC .

Proof. Through A draw $AL \parallel$ to BD , and join AD , FC .

Since $\angle BAG$ is a rt. $\angle$, (Cons.)

and $\angle BAC$ is a rt. $\angle$, (Hyp.)

$\therefore CAG$ is a st. line. (52)

$\therefore$ sq. BG is double the $\triangle FBC$,
having the same base and between the same $\parallel$ s (368),

and rect. BL is double the $\triangle ABD$.

Again, since $FB = AB$, and $BC = BD$, (Cons.)

and $\angle FBC = \angle ABD$,

each being the sum of a rt. $\angle$ and the common $\angle ABC$,

$\therefore \triangle FBC = \triangle ABD$,

having two sides and the included $\angle$ equal, each to each (104).

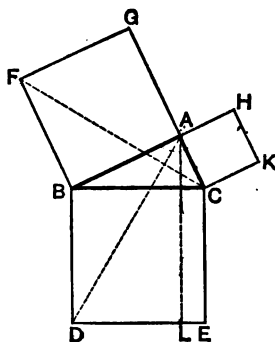
$\therefore$ sq. BG = rect. BL . (Ax. 6)

In like manner, by joining BK , AE , it may be proved
that sq. HC = rect. CL .

$\therefore$ whole sq. $BDEC$ = sum of sqs. BG and HC . (Ax. 1)

$\therefore$ sq. on BC = sq. on AB + sq. on AC . Q.E.D.

NOTE.—This proposition is commonly called the *Pythagorean Proposition*, because it is said to have been discovered by Pythagoras (born about 600 B.C.). The above demonstration of it was given by Euclid, about 300 B.C. (Prop. 47, Book I. Euclid).



Proposition 8. Theorem.

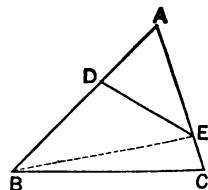
375. *The areas of two triangles having an angle of the one equal to an angle of the other, are to each other as the products of the sides including the equal angles.*

Hyp. Let $\triangle ABC$, $\triangle ADE$ be the two $\triangle$ s having the common $\angle A$.

$$\text{To prove} \quad \frac{\triangle ABC}{\triangle ADE} = \frac{AB \times AC}{AD \times AE}.$$

Proof. Join BE .

Since the $\triangle$ s $\triangle ABC$, $\triangle ABE$ have their bases AC , AE in the same straight line, and the common vertex B , they have the same altitude.



$$\therefore \frac{\triangle ABC}{\triangle ABE} = \frac{AC}{AE},$$

$\triangle$ s of the same altitude are to each other as their bases (369).

Also, the $\triangle$ s $\triangle ABE$, $\triangle ADE$ have the same altitude, since their bases AB , AD , opp. the common vertex E , are in the same straight line.

$$\therefore \frac{\triangle ABE}{\triangle ADE} = \frac{AB}{AD}. \quad (369)$$

Multiplying these equalities, we have

$$\frac{\triangle ABC}{\triangle ADE} = \frac{AB \times AC}{AD \times AE}. \quad (290)$$

Q. E. D.

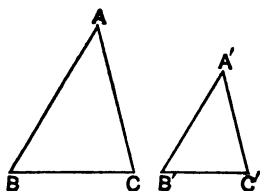
376. COR. *If the products of the sides including the equal angles are equal, the triangles are equivalent.*

EXERCISE.

If two triangles have an angle of the one supplementary to an angle of the other, their areas are to each other as the products of the sides including these angles.

Proposition 9. Theorem.

377. *The areas of similar triangles are to each other as the squares of their homologous sides.*



Hyp. Let $ABC, A'B'C'$ be similar Δs .

To prove
$$\frac{\Delta ABC}{\Delta A'B'C'} = \frac{\overline{BC}^2}{\overline{B'C'}^2}.$$

Proof. Since $\angle B = \angle B'$, (Hyp.)

$$\therefore \frac{\Delta ABC}{\Delta A'B'C'} = \frac{BA \times BC}{B'A' \times B'C'} = \frac{BA}{B'A'} \times \frac{BC}{B'C'}. \quad (375)$$

But
$$\frac{BA}{B'A'} = \frac{BC}{B'C'},$$

homologous sides of similar Δs are proportional (307).

Substitute in the above equality for $\frac{BA}{B'A'}$, its equal $\frac{BC}{B'C'}$.

$$\therefore \frac{\Delta ABC}{\Delta A'B'C'} = \frac{BC}{B'C'} \times \frac{BC}{B'C'} = \frac{\overline{BC}^2}{\overline{B'C'}^2}. \quad \text{Q.E.D.}$$

378. *Cor. The areas of two similar triangles are to each other as the squares of any two homologous lines.*

EXERCISES.

1. ABC is a triangle. AE and BF , intersecting in G , are drawn to bisect the sides BC, AC in E and F . Compare the areas of the triangles AGB, FGE .

2. The base and altitude of a triangle are 18 and 8 respectively; what is the altitude of a similar triangle whose base is 12?

3. A has a triangular piece of ground, the base of the triangle being 40 rods; what is the base of a similarly-shaped lot containing 4 times as much? *Ans.* 80 rods.

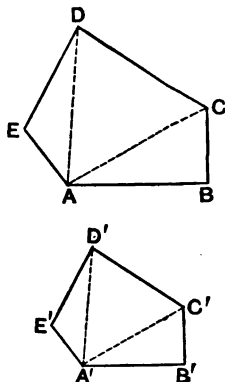
Proposition 10. Theorem.

379. *The areas of similar polygons are to each other as the squares of their homologous sides.*

Hyp. Let S and S' denote the areas of the two similar polygons $ABCDE$ and $A'B'C'D'E'$, in which AB and $A'B'$ are homologous sides.

$$\text{To prove} \quad \frac{S}{S'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Proof. Join $AC, AD, A'C', A'D'$, dividing the two polygons into the same number of similar $\triangle$ s, and similarly placed. (321)



$$\therefore \frac{\triangle ABC}{\triangle A'B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

$$\text{Also, } \frac{\triangle ACD}{\triangle A'C'D'} = \frac{\overline{CD}^2}{\overline{C'D'}^2}, \text{ and } \frac{\triangle ADE}{\triangle A'D'E'} = \frac{\overline{DE}^2}{\overline{D'E'}^2},$$

similar $\triangle$ s are to each other as the squares of their homologous sides (377).

$$\text{But } \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \text{etc.} \quad (307)$$

$$\therefore \frac{\triangle ABC}{\triangle A'B'C'} = \frac{\triangle ACD}{\triangle A'C'D'} = \frac{\triangle ADE}{\triangle A'D'E'}. \quad (\text{Ax. 1})$$

$$\begin{aligned} \therefore \frac{\triangle ABC + \triangle ACD + \triangle ADE}{\triangle A'B'C' + \triangle A'C'D' + \triangle A'D'E'} &= \frac{\triangle ABC}{\triangle A'B'C'} \\ &= \frac{\overline{AB}^2}{\overline{A'B'}^2}. \end{aligned} \quad (296)$$

$$\therefore \frac{S}{S'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Q.E.D.

380. COR. 1. *The areas of similar polygons are to each other as the squares of any two homologous lines.*

381. COR. 2. *The homologous sides of similar polygons are to each other as the square roots of their areas.*

PROBLEMS OF CONSTRUCTION.

Proposition 11. Problem.

382. *To construct a triangle equivalent to a given polygon.*

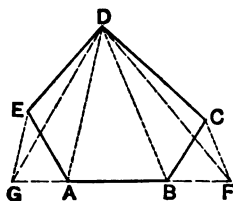
Given, the polygon ABCDE.

Required, to construct a $\triangle$ equal to it in area.

Cons. Join BD.

From C draw CF $\parallel$ to DB, meeting AB produced in F; join DF.

In like manner join AD, draw EG $\parallel$ to DA, meeting BA produced in G, and join DG.



Then $\text{area DGF} = \text{area ABCDE}.$

Proof. $\triangle BDF = \triangle BDC,$

having the same base BD and their vertices C and F in the st. line CF $\parallel$ to the base (367).

Adding the polygon ABDE to each of these equalities, we have

$$\text{area AFDE} = \text{area ABCDE}. \quad (\text{Ax. 2})$$

Again, $\triangle ADG = \triangle ADE.$

Adding the $\triangle ADF$ to each, we have

$$\text{area GDF} = \text{area AFDE} = \text{area ABCDE}. \quad \text{Q.E.F.}$$

383. SCH. In the same manner a triangle may be found equivalent to a polygon having any number of sides, since each operation of the above process diminishes the number of sides of the polygon by one.

Proposition 12. Problem.

384. *To construct a square equivalent to a given parallelogram.*

Given, a $\square$ ABCD, CE its altitude.

Required, to construct a square of the same area.

Cons. Find a mean proportional between AB and CE (349), and represent it by FG.

The square described upon the line FG will be equivalent to the $\square$ ABCD.

Proof. Since $AB : FG = FG : CE$, (Cons.)

$$\therefore \overline{FG}^2 = AB \times CE. \quad (281)$$

$$\therefore \text{area } FGHK = \text{area } ABCD. \quad \text{Q. E. F.}$$

385. COR. 1. *To construct a square equivalent to a given triangle, we take for its side a mean proportional between the base of the triangle and one-half its altitude.*

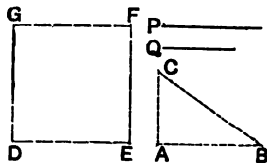
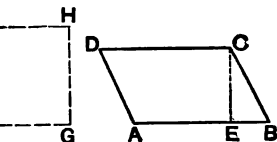
386. COR. 2. *A square can be found equivalent to any given polygon, by first constructing a triangle equivalent to the given polygon (382), and then constructing a square equivalent to the triangle (385).*

Proposition 13. Problem.

387. *To construct a square equivalent to the sum of two given squares.*

Given, P and Q the sides of the squares.

Required, to construct a square equivalent to their sum.



Cons. Draw $AB = P$.

At A construct the rt. $\angle A$, draw $AC = Q$, and join BC.

Construct the square DEFG, with each of its sides equal to BC.

Then DEFG is the square required.

Proof. $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$. (374)

$$\therefore \overline{DE}^2 = P^2 + Q^2. \quad \text{Q.E.F.}$$

388. SCH. In the same way a square may be found equivalent to the sum of any number of squares; for the same construction which reduces two of them to one will reduce three of them to two, and these two to one; and so of others.

Proposition 14. Problem.

389. To construct a square equivalent to the difference of two given squares.

Given, P and Q the sides of the squares.

Required, to construct a square equivalent to their difference.

Cons. Draw the indefinite st. line AH.

At A construct the rt. $\angle A$, draw $AC = Q$, the shorter side of the given lines.

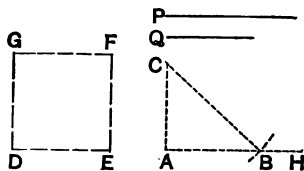
With centre C, and a radius = P, describe an arc cutting AH in B.

Construct the square DEFG, with each of its sides equal to AB.

Then DEFG is the square required.

Proof. $\overline{AB}^2 = \overline{BC}^2 - \overline{AC}^2$. (374)

$$\therefore \overline{DE}^2 = P^2 - Q^2. \quad \text{Q.E.F.}$$

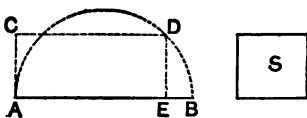


Proposition 15. Problem.

390. To construct a rectangle equivalent to a given square, and having the sum of two adjacent sides equal to a given line.

Given, the square S , and the line $AB =$ the sum of the sides.

Required, to construct a rectangle $= S$, having the sum of its base and altitude $= AB$.



Cons. Upon AB as a diameter, describe a semicircle.

At A erect $AC \perp$ to AB and $=$ a side of S .

Draw $CD \parallel$ to AB , cutting the $\odot$ at D , and draw $DE \perp$ to AB .

Then AE and EB are the base and altitude of the required rectangle.

Proof. $AE \times EB = \overline{DE}^2$,

since DE is a mean proportional between AE and EB (325),

and $DE = AC =$ a side of S . (Cons.)

$\therefore AE \times EB = S$. Q. E. F.

391. SCH. When the side of the square S exceeds half the line AB the problem is impossible.

EXERCISES.

1. Construct a square equivalent to the sum of two squares whose sides are 6 and 8 inches.

2. Construct a square equivalent to the difference of two squares whose sides are 15 and 25 feet.

3. The perimeter of a rectangle is 144 feet, and the length is three times the altitude: find the area.

4. On a given straight line construct a triangle equal to a given triangle and having its vertex on a given straight line not parallel to the base.

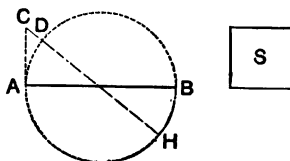
5. Construct a parallelogram that shall be equal in area and perimeter to a given triangle.

Proposition 16. Problem.

392. *To construct a rectangle equivalent to a given square, and having the difference of two adjacent sides equal to a given line.*

Given, the square S , and the line $AB =$ the difference of the sides.

Required, to construct a rectangle $= S$, having the difference of its base and altitude $= AB$.



Cons. Upon AB as a diameter, describe a $\odot$.

At A erect $AC \perp$ to AB and $=$ a side of S .

Through C and the centre of the $\odot$ draw CH , cutting the $\odot$ at D and H .

Then CH and CD are the base and altitude of the required rectangle.

Proof. $CH \times CD = \overline{AC}^2$,

a tangent is a mean proportional between the whole secant and the ext. seg. (339).

$$\therefore CH \times CD = S,$$

and the difference between CH and CD is $DH = AB$.

$\therefore CH \times CD$ is the required rectangle.

Q. E. F.

EXERCISES.

1. The bases of a trapezoid are 14 and 16 feet ; the non-parallel sides are each 6 feet: find the area of the trapezoid.

2. Construct a rhombus equal to a given parallelogram and having one of the sides of the parallelogram for one side of the rhombus.

Proposition 17. Problem.

393. *Two similar polygons being given, to construct a similar polygon equal to their sum.*

Given, two homologous sides P and Q of two similar polygons R and S .

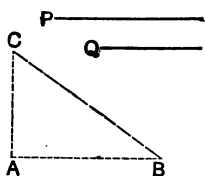
Required, to construct a similar polygon equivalent to their sum.

Cons. Draw $AB = P$.

At A construct the rt. $\angle A$, draw $AC = Q$, and join BC .

On BC , homologous to P and Q , construct a polygon T similar to R and S , as in (354).

Then T is the polygon required.



$$\text{Proof.} \quad \frac{R}{T} = \frac{P^2}{BC^2}, \text{ and } \frac{S}{T} = \frac{Q^2}{BC^2}. \quad (379)$$

$$\text{Adding,} \quad \frac{R + S}{T} = \frac{P^2 + Q^2}{BC^2}.$$

$$\text{But} \quad \overline{BC^2} = P^2 + Q^2. \quad (\text{Cons.})$$

$$\therefore T = R + S. \quad \text{Q.E.D.}$$

394. SCH. To construct a polygon similar to two given similar polygons, and equivalent to their *difference*, we find the side of a square equivalent to the difference of the squares on P and Q (389), and on this side, homologous to P and Q , construct a polygon similar to the given polygons R and S (354). This will be the polygon required (379).

Proposition 18. Problem.

395. *To find two straight lines proportional to two given polygons.*

Given, two polygons R and S.

Required, to find two st. lines proportional to R and S.

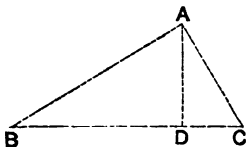
Cons. Find two squares equivalent to the given polygons R and S (386); let P and Q be the sides of these squares.

Construct the rt. $\angle$ A, draw AB = P, and AC = Q.

Join BC, and draw AD $\perp$ to BC.

Then BD and DC are the lines required.

Proof. Since AD is a $\perp$ from the rt. $\angle$ A on the hypotenuse BC,



$$\therefore \overline{AB}^2 : \overline{AC}^2 = BD : DC,$$

the sqs. of the sides about the rt. $\angle$ are proportional to the adj. segments of the hypotenuse (324).

But $AB = P$, and $AC = Q$. (Cons.)

$$\therefore P^2 : Q^2 = BD : DC,$$

and $\therefore$ BD, DC are proportional to the areas of the given polygons. Q. E. F.

EXERCISE.

Bisect a quadrilateral by a straight line drawn from one of its vertices.

Let ABCD be the quad.; bisect BD in E, let E lie between AC and B; through E draw EF $\parallel$ to AC to meet BC in F; join CE, EA, AF; then AFCD = $\frac{1}{2}$ ABCD.

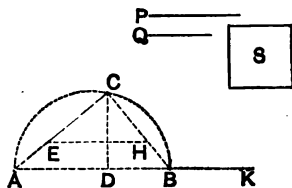
Proposition 19. Problem.

396. To construct a square which shall have to a given square the ratio of two given lines.

Given, the square S , and the ratio $P : Q$.

Required, to construct a square which shall be to S as P is to Q .

Cons. Draw the st. line AK ; take $AD = P$ and $DB = Q$.



Upon AB as a diameter, describe a $\frac{1}{2}\odot$ ce.

At D erect the $\perp$ DC , and join AC , BC .

On CB , or CB produced, take $CH =$ a side of S .

Draw $HE =$ to BA .

Then CE is the side of the required square.

Proof. Since CD is a $\perp$ from the rt. $\angle C$ to the hypotenuse AB ,

$$\therefore \overline{CA}^2 : \overline{CB}^2 = AD : DB. \quad (324)$$

But

$$CA : CB = CE : CH,$$

a st. line $\parallel$ to a side of a $\triangle$ cuts the other two sides proportionally (298),

and

$$\overline{CA}^2 : \overline{CB}^2 = \overline{CE}^2 : \overline{CH}^2. \quad (295)$$

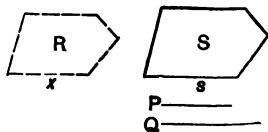
$$\therefore \overline{CE}^2 : \overline{CH}^2 = AD : DB = P : Q.$$

But

$$\overline{CH}^2 = S. \quad (\text{Cons.})$$

$$\therefore \overline{CE}^2 : S = P : Q. \quad \text{Q.E.F.}$$

397. SCH. To construct a polygon similar to a given polygon S , and having to it the given ratio of P to Q , we find, as in (396), a side x so that x^2 shall be to s^2 (where s is a side of S) as P is to Q , and upon x as a side homologous to s , construct the polygon R similar to S (354); this will be the polygon required.



Proposition 20. Problem.

398. *To construct a polygon equivalent to a given polygon P, and similar to a given polygon Q.*

Given, two polygons P and Q.

Required, to construct a polygon equivalent to P and similar to Q.

Cons. Find p and q , the sides of squares equivalent respectively to P and Q. (386)

Take any side of Q as AB, and find a fourth proportional $A'B'$ to q , p , and AB. (347)

Upon $A'B'$, homologous to AB, construct Q' similar to Q. (354)

Then Q' is the polygon required.

Proof. Since Q' is similar to Q, (Cons.)

$$\therefore Q : Q' = \overline{AB}^2 : \overline{A'B'}^2. \quad (379)$$

But $q : p = AB : A'B'. \quad (Cons.)$

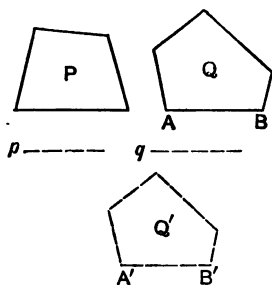
$$\therefore Q : Q' = q^2 : p^2. \quad (Ax. 1)$$

But $P = p^2$, and $Q = q^2. \quad (Cons.)$

$$\therefore Q : Q' = Q : P; \text{ and } \therefore Q' = P.$$

$$\therefore Q' \text{ is equivalent to P and similar to Q.}$$

Q. E. F.



APPLICATIONS.

1. To find the area of an equilateral triangle in terms of its side a .*

Let h denote the alt., and S the area, of the Δ .

$$\text{Then} \quad h = \sqrt{a^2 - \frac{a^2}{4}} = \frac{a}{2} \sqrt{3}.$$

$$\text{But (366),} \quad S = \frac{a \times h}{2}.$$

$$\therefore S = \frac{a^2 \sqrt{3}}{4}.$$

2. To find the area of a triangle in terms of its sides and the radius of the circumscribing circle.

Let a, b, c denote the sides and h the alt. of the Δ , and R the radius of the circumscribing $\odot$.

$$\text{By Ex. 4 (354),} \quad bc = 2Rh.$$

$$\therefore abc = 2Rah = 4RS. \quad (366)$$

$$\therefore S = \frac{abc}{4R}.$$

Therefore, the area of a triangle is equal to the product of the three sides divided by four times the radius of the circumscribing circle.

3. To find the area of a triangle in terms of its sides.

$$\text{By Ex. 1 (354),} \quad h = \frac{2}{a} \sqrt{s(s-a)(s-b)(s-c)}.$$

$$\text{And since} \quad S = \frac{1}{2}ah,$$

$$\therefore S = \sqrt{s(s-a)(s-b)(s-c)}.$$

5. Find the area of an equilateral triangle if a side = 1 foot. Ans. 0.433 sq. ft.

6. Find (1) the area of the triangle whose sides are 3, 4, and 5 feet, and (2) the radius of the circumscribing circle.

Ans. (1) 6 sq. ft. ; (2) 2.5 ft.

EXERCISES.

THEOREMS.

1. A triangle is divided by each of its medians into two parts of equal area.

2. A parallelogram is divided by its diagonals into four triangles of equal area.

3. ABC is a triangle, and its base BC is bisected at D ; if H be any point in the median AD , show that the triangles ABH , ACH are equal in area.

4. In AC , a diagonal of the parallelogram $ABCD$, any point H is taken, and HB , HD are drawn: show that the triangle BAH is equal to the triangle DAH .

5. If two triangles have two sides of one respectively equal to two sides of the other, and the included angles supplementary; show that the triangles are equal in area.

6. $ABCD$ is a parallelogram, and E , H are the middle points of the sides AD , BC ; if Z is any point in EH , or EH produced, show that the triangle AZB is one quarter of the parallelogram $ABCD$.

7. If $ABCD$ is a parallelogram, and E , H any points in DC and AD respectively, show that the triangles AEB , BHC are equal in area.

8. $ABCD$ is a parallelogram, and P is any point within it: show that the sum of the triangles PAB , PCD is equal to half the parallelogram.

9. Find the ratio of a rectangle 18 yards by $14\frac{1}{2}$ yards to a square whose perimeter is 100 feet.

10. The medians AD , BE of the triangle ABC are produced to meet the straight line, drawn through C parallel to AB , in F , G respectively: prove that the triangle AGC is equal in area to the triangle BCF .

$AD = DF$ (105), etc.

11. $ABCD$ is a parallelogram; through A any two straight lines AE , AF are drawn meeting BC in E and CD in F ;

through D, DG is drawn parallel to AE meeting AF in G. Show that the parallelogram having AE, AG as adjacent sides is equal to ABCD.

Draw EH $\parallel$ to AG meeting DG produced in H; produce BC to meet DH in K; $\square AH = \square AK$; why? etc.

12. If O is the centroid (172) of the triangle ABC, prove that the triangles AOB, BOC, COA are equal in area.

13. Show that the line joining the middle points of the parallel sides of a trapezoid divides the area into two equal parts.

14. If any point in one side of a triangle be joined to the middle points of the other sides, the area of the quadrilateral thus formed is one-half that of the triangle.

15. Prove that any straight line which bisects a parallelogram must pass through the intersection of its diagonals.

Let EF bisect $\square ABCD$ meeting AB in E, DC in F, BD in O; join ED, FB, etc.

16. APB, ADQ are two straight lines such that the triangles PAQ, BAD are equal. If the parallelogram ABCD be completed, and BQ joined cutting CD in R, show that $CR = AP$.

17. If the straight line joining the middle points of two opposite sides of any quadrilateral divide the area into two equal parts; show that the two bisected sides are parallel.

18. Through the vertices of a quadrilateral straight lines are drawn parallel to the diagonals: prove that the figure thus formed is a parallelogram which is double the quadrilateral.

19. Through D, E, the middle points of the sides BC, CA of a triangle ABC, any two parallel straight lines are drawn meeting AB or AB produced in F and G: prove that the parallelogram DEGF is half the triangle ABC.

20. In a trapezoid the straight lines, drawn from the middle point of one of the non-parallel sides to the ends of the opposite side, form with that side a triangle equal to half the trapezoid.

21. Prove that the points F, A, K, in the figure of Prop. 7, are collinear.

22. In the figure of Prop. 7, if FG , KH be produced to meet in P , prove that PA produced cuts BC at right angles.

23. Points E , F , G , H are taken respectively in the sides AB , BC , CD , DA of a rectangle $ABCD$; if $EF = GH$, prove that

$$\overline{AG}^2 + \overline{CH}^2 = \overline{AF}^2 + \overline{CE}^2.$$

24. From D , the middle point of the side AC of a right triangle ABC , DE is drawn perpendicular to the hypotenuse AB : prove that $\overline{BE}^2 = \overline{AE}^2 + \overline{BC}^2$.

25. $ABCD$ is a parallelogram. If AC is bisected in O and a straight line MON is drawn to meet AB , CD in M , N respectively, and OR parallel to AB meets AN in R ; prove that the triangles ARM , CRN are equal.

$NC = AM$ (106), $\therefore AN$ is $\parallel$ to CM (133), etc.

26. If through the vertices of a triangle ABC there be drawn three parallel straight lines AD , BE , CF to meet the opposite side or sides produced in D , E , F : prove that the area of the triangle DEF is double that of ABC .

Let D be in BC ; E , F in CA , BA produced; $\triangle EFB = \triangle ECB$, etc.

27. ABC , DEF are triangles having the angles A and D equal, and AB equal to DE : show that the triangles are to each other as AC to DF .

28. The side BC of the triangle ABC is bisected in D : prove that any straight line through D divides the sides AB , AC into segments which are proportional.

29. On the sides AB , AC of a triangle ABC points D , E are taken such that AD is to DB as CE is to EA : if the lines CD , BE intersect in F , prove that the triangle BFC is equal to the quadrilateral $ADFE$.

30. $ABCD$ is a square. A line drawn through A cuts the sides BC , CD , produced if necessary, in E , F . Prove that the triangle CEF is to the triangle ABC as the difference of the lines CE and CF is to BC .

31. O is a point inside a triangle ABC ; AO , BO , CO produced meet the sides in D , E , F , respectively. If AO is to OD as BO to EO as CO to OF , prove that O is the centroid (172) of the triangle ABC .

32. ABC is a triangle, and points D, E are taken in BC, CA respectively, such that $BD = \frac{1}{3}BC$ and $AE = \frac{1}{3}AC$; AD, BE intersect in O. Prove that $\triangle AOB = \frac{1}{3} \triangle ABC$.

$\triangle BAD : \triangle ADC = BD : DC = 1 : 2$; sim., $\triangle OBD : \triangle ODC = 1 : 2$, etc.

33. ABCD is a quadrilateral having two of its sides AB, CD parallel; AF, CG are drawn parallel to each other meeting BC, AD, respectively, in F, G. Prove that BG is parallel to DF.

Produce CB, DA to meet in E, etc.

34. ABC is a triangle, G its centroid, D the middle point of BC, AE its perpendicular to BC. Show that if the rectangle of which AE, EC are adjacent sides be completed, the fourth corner being F, then FG produced bisects BE.

Produce FG to meet BC in H; $HD : AF = 1 : 2$, etc.

35. ABCD, AB'C'D' are two squares, BAB', DAD' being straight lines; B'C meets AD in E, and C'D meets AB' in F. Prove that $AE = AF$.

36. A triangle ABC has the side $AB = 2AC$; from C is drawn CD to a point D in AB such that $\angle ACD = \angle ABC$: show that $\triangle BCD = 3 \triangle ACD$.

37. If in an isosceles triangle ABC the two equal sides AB, CA be divided at F and E respectively, in any given ratio; the straight line FE will meet BC produced at a point D, such that $CD : BD = \overline{AF}^2 : \overline{FB}^2$, or $= \overline{CE}^2 : \overline{EA}^2$.

Draw $AG \parallel$ to CB meeting EF produced in G, etc.

38. ABC is an isosceles triangle having the sides AB, AC equal, and is such that if BD be drawn bisecting the angle ABC and cutting AC in D, then AD is equal to BC. Show that the angle C is double the angle A.

39. Perpendiculars are drawn from the vertices of a triangle on any straight line through the centroid of the triangle: prove that one of these perpendiculars is equal to the sum of the other two.

40. Perpendiculars are drawn from the vertices and the centroid of a triangle on a given straight line: prove that the perpendicular from the centroid is the arithmetic mean of the perpendiculars from the vertices.

Let ABC be the $\triangle$, O its centroid, D the mid. pt. of BC; draw AL, BM, CN, DR, OK $\perp$ to given line; bisect AO in E; draw EH $\perp$ to LM, etc.

41. If the vertical angle C of a triangle ABC be bisected by a line which meets the base in D, and is produced to a point E, so that the rectangle of CD and CE is equal to that of AC and CB; show that if the base and vertical angle be constant the position of E is fixed.

Since $CD \times CE = AC \times CB$, $\therefore CD : AC = BC : CE$, and $\angle ACD = \angle BCD$, $\therefore \angle CAB = \angle CEB$, etc.

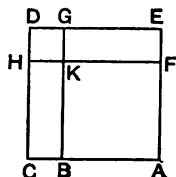
42. ABC is a triangle inscribed in a circle; from A straight lines AD, AE are drawn parallel to the tangents at B, C respectively, meeting BC, produced if necessary, in D, E: prove that BD is to CE as the square on AB is to the square on AC.

Let BF, CG be the tangs. : $\angle ADB = \text{alt.}$ $\angle DBF = \angle BAC$ (238 and 243), and $\angle B$ is common to both $\triangle s$ BAC, BAD, etc.

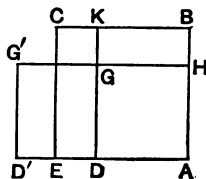
43. If P be a point on a diameter AB of a circle, and PT be the perpendicular on the tangent at a point Q; show that $PT \times AB = AP \times PB + PQ^2$.

Produce QP to meet the $\odot$ in R; draw diam. RS; $\angle TQP = \angle QSR$, etc.

44. Prove that the square constructed on the sum of two straight lines is equivalent to the sum of the squares constructed on the two lines, together with twice the rectangle of the lines.



45. Prove that the square constructed on the difference of two straight lines is equivalent to the sum of the squares constructed on the lines, diminished by twice the rectangle of the lines.



46. Prove that the rectangle of the sum and the difference of two straight lines is equivalent to the difference of the squares constructed on the lines.

47. ABC is an isosceles triangle, CA, CB being the equal sides; BO is drawn at right angles to BC to meet CA produced in O: show that the square on OB is equal to the square on OA with twice the rectangle OA, AC.

48. If a straight line be divided into two equal and

also into two unequal parts, the rectangle of the unequal parts with the square on the line between the points of section, is equal to the square on half the line. (Euclid, B. II, prop. 5).

49. The square on the straight line, drawn from the vertex of an isosceles triangle to any point in the base, is less than the square on a side of the triangle by the rectangle of the segments of the base.

50. The sides AB, CD of a quadrilateral ABCD inscribed in a circle are produced to meet in P; and PE, PF are drawn perpendicular to AD, BC respectively: prove that AE is to ED as CF to FB.

51. The sides AB, AD, produced if necessary, of a parallelogram ABCD, meet a line through C in E, F respectively; CB, CD, produced if necessary, meet a line through A in G, H respectively. Prove that GE is parallel to HF.

PROBLEMS.

52. Construct a square equal to three given squares.

53. Construct a square which shall be five times as great as a given square.

54. Construct a triangle equal in area to a given quadrilateral.

55. Construct an isosceles triangle equal in area to a given triangle and having a given vertical angle.

56. Divide a straight line into two parts, so that the sum of the squares on the parts may be equal to a given square.

57. Trisect a triangle by straight lines drawn from a given point in one of its sides.

58. Find a point O inside a triangle ABC such that the triangles OAB, OBC, OCA are equal.

59. Construct a square that shall be one-third of a given square.

60. Divide a straight line into two parts so that the rectangle contained by the whole and one part may be equal to a given square.

Let AB be the given line. Draw $BD \perp$ to AB so that $\overline{BD}^2 = \text{given square}$: join AD, draw $DE \perp$ to AD, meeting AB produced in E.

61. Produce AB to C so that the rectangle of AB and AC may be equal to a given square.

62. Construct a parallelogram equal to a given triangle and having one of its angles equal to a given angle.

63. Given three similar triangles: construct another similar triangle and equivalent to their sum. (377).

64. Construct a triangle similar to a given triangle ABC which shall be to ABC in the ratio of AB to BC.

Find DE a mean proportional between AB, BC, etc.

65. Bisect a triangle by a straight line parallel to one of its sides.

Bisect AC of the $\triangle ABC$ in D; from AC cut off AE a mean proportional to AC, AD, etc.

66. A is a point on a given circle: draw through A a straight line PAQ meeting the circle in P and a given straight line in Q, so that the ratio of PA to AQ may be equal to a given ratio.

Draw AB to any pt. B in the given line MN: produce BA to C so that BA : AC = given ratio, etc.

67. From the vertex of a triangle draw a line to the base, so that it may be a mean proportional between the segments of the base.

About the given $\triangle$ describe a $\odot$, etc.

68. Show how to draw through a given point in a side of a triangle a straight line dividing the triangle in a given ratio.

69. Construct a triangle equal to a given triangle and having one of its angles equal to an angle of the triangle, and the sides containing this angle in a given ratio.

70. Draw through a given point a straight line, so that the part of it intercepted between a given straight line and a given circle may be divided at the given point in a given ratio.

Let P be the given pt., XY the given line, A the cent. of given $\odot$: produce AP to meet XY in Q; in PA take B so that QP : PB = given ratio, etc.

Book V.

REGULAR POLYGONS. THE CIRCLE. MAXIMA AND MINIMA.

REGULAR POLYGONS.

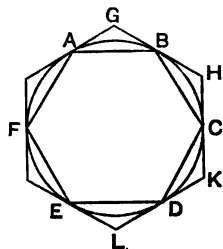
399. DEF. A *regular polygon* is a polygon which is both equilateral and equiangular; as, for example, the equilateral triangle and the square.

Proposition 1. Theorem.

400. *If the circumference of a circle be divided into any number of equal arcs, (1) the chords of these arcs form a regular polygon inscribed in the circle, and (2) the tangents at the points of division form a regular polygon circumscribed about the circle.*

Hyp. Let the $\odot$ ce be divided into equal arcs at the pts. A, B, C, etc.; let AB, BC, CD, etc., be chords of these arcs, and GBH, HCK, etc., be tangents.

(1) *To prove* that ABCD . . . is a regular polygon.



Proof. Since arc AB = arc BC = arc CD = etc., (Hyp.)
 $\therefore$ chd. AB = chd. BC = chd. CD = etc.,
in the same $\odot$ equal arcs are subtended by equal chords (194);
 and $\angle$ FAB = $\angle$ ABC = $\angle$ BCD = etc.,
all $\angle$ s inscribed in equal segments are equal (239).
 $\therefore$ ABCDEF is a regular polygon. (399)

(2) *To prove* that GHK . . . is a regular polygon.

Proof. In the $\triangle$ s ABG, BCH, etc.,
 AB = BC = CD = etc.,
 and $\angle$ GAB = $\angle$ GBA = $\angle$ HBC = $\angle$ HCB = etc.,
being measured by halves of equal arcs (243).
 $\therefore$ the $\triangle$ s ABG, BCH, etc., are all equal and isosceles.

$$\therefore AG = BG = BH = CH = \text{etc.},$$

$$\text{and} \quad \angle G = \angle H = \angle K = \text{etc.}$$

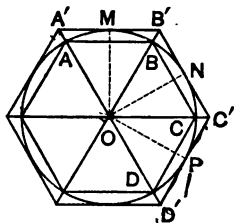
$$\therefore GH = HK = \text{etc.},$$

$$\text{and} \quad \therefore GHKL \dots \text{ is a regular polygon.} \quad (399)$$

Q. E. D.

401. COR. 1. *If a regular inscribed polygon is given, the tangents at the vertices of the given polygon form a regular circumscribed polygon of the same number of sides.*

402. COR. 2. *If a regular inscribed polygon $ABCD \dots$ is given, the tangents at the middle points $M, N, P, \text{etc.}$, of the arcs $AB, BC, CD, \text{etc.}$, form a regular circumscribed polygon whose sides are parallel to those of the inscribed polygon, and whose vertices $A', B', C', \text{etc.}$, lie on the radii $OAA', OBB', \text{etc.}$ For, the sides $AB, A'B'$ are $\parallel$, being $\perp$ to OM , (204) and (210), and the same for the others; also, since $B'M = B'N$ (268), the pt. B' must lie on the bisector OB (160) of the $\angle MON$.*



403. COR. 3. *If the chords $AM, MB, BN, \text{etc.}$, be drawn, the chords form a regular inscribed polygon of double the number of sides of $ABCD \dots$.*

404. COR. 4. *If through the points $A, B, C, \text{etc.}$, tangents are drawn intersecting the tangents $A'B', B'C', \text{etc.}$, a regular circumscribed polygon is formed of double the number of sides of $A'B'C'D' \dots$.*

405. SCH. It is clear that the area of any inscribed polygon is less than the area of the inscribed polygon of double the number of sides; and the area of a circumscribed polygon is greater than that of the circumscribed polygon of double the number of sides.

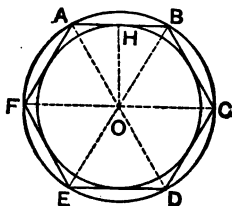
Proposition 2. Theorem.

406. *A circle may be circumscribed about a regular polygon, and a circle may be inscribed in it.*

Hyp. Let ABCDEF be a regular polygon.

(1) *To prove that a $\odot$ may be circumscribed about it.*

Proof. Bisect the $\angle$ s A and B, and let the bisectors meet at O.



Since $\angle FAB = \angle ABC$, (Hyp.)

$\therefore \angle OAB = \angle OBA$, (Ax. 7)

and $\therefore OA = OB$.

Join OC, OD.

In the $\triangle$ s OAB and OBC,

$AB = BC$ (399), OB is common,

and $\angle OBA = \angle OBC$. (Cons.)

$\therefore \triangle OBA = \triangle OBC$. (104)

$\therefore \angle OAB = \angle OCB = \frac{1}{2} \angle BCD$,

and $OA = OC$.

In the same way it may be shown that each $\angle$ of the

polygon is bisected by the line joining it with the pt. O, and that $OA = OB = OC = OD = \text{etc.}$

$\therefore$ the $\odot$ described with centre O and radius OA will pass through each of the angular points.

(2) To prove that a $\odot$ may be inscribed in ABCDEF.

Proof. Since the sides AB, BC, etc., are equal chords of the circumscribed circle, they are equally distant from the centre. (206)

Therefore, if a circle be described with the centre O and the perpendicular OH as a radius, this circle will be inscribed in the polygon: Q. E. D.

407. DEFS. The point O, which is the common centre of the inscribed and circumscribed circles, is called the **centre** of the regular polygon.

The **radius** of a regular polygon is the radius OA of the circumscribed circle.

The **apothem** of a regular polygon is the radius OH of the inscribed circle.

The **angle at the centre** of a regular polygon is the angle between two radii drawn to the extremities of any side, as AOB.

408. COR. 1. *The inscribed and circumscribed circles of a regular polygon are concentric.*

409. COR. 2. *The perpendicular bisectors of the sides of a regular polygon all pass through its centre.*

410. COR. 3. *The radius drawn to any vertex of a regular polygon bisects the angle at the vertex.*

411. COR. 4. *If lines be drawn from the centre of a regular polygon to each of its vertices, the polygon will be divided into as many equal isosceles triangles as it has sides.*

412. COR. 5. *Each angle at the centre of a regular polygon is equal to four right angles divided by the number of sides of the polygon.*

413. COR. 6. *The interior angle of a regular polygon is the supplement of the angle at the centre.*

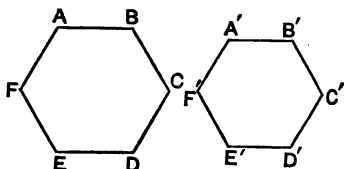
Proposition 3. Theorem.

414. *Regular polygons of the same number of sides are similar.*

Hyp. Let P and P' be two regular polygons of the same number of sides.

To prove that P and P' are similar.

Proof. Since the polygons are regular,



$$\therefore AB = BC = CD = \text{etc.},$$

$$\text{and} \quad A'B' = B'C' = C'D' = \text{etc.} \quad (399)$$

$$\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \text{etc.}$$

Also, if the polygons have each n sides, the sum of all the int. $\angle$ s of each polygon is $(2n - 4)$ rt. $\angle$ s. (148)

Since the polygons are equiangular (399), and have each the same number of sides, (Hyp.)

$$\therefore \text{each } \angle \text{ of each polygon} = \frac{2n - 4}{n} \text{ rt. } \angle \text{s.} \quad (150)$$

$$\therefore \angle A = \angle A', \angle B = \angle B', \angle C = \angle C', \text{ etc.}$$

Hence the polygons are mutually equiangular and have their homologous* sides proportional.

$$\therefore \text{the polygons are similar.} \quad (307)$$

Q.E.D.

415. COR. 1. *The perimeters of regular polygons of the same number of sides are to each other as any two homologous* sides.* (322)

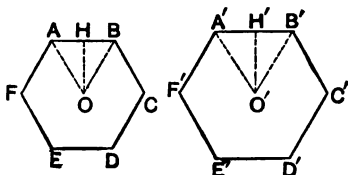
416. COR. 2. *The areas of regular polygons of the same number of sides are to each other as the squares of any two homologous* sides.* (379)

* Since the polygons are regular, any side of one may be taken as homologous to any side of the other.

Proposition 4. Theorem.

417. *The perimeters of any two regular polygons of the same number of sides are to each other as the radii of their circumscribed circles, or as the radii of their inscribed circles.*

Hyp. Let P and P' be the perimeters of two regular polygons, O and O' their centres, OA and $O'A'$ the radii of their circumscribed $\odot$ s, OH and $O'H'$ the radii of their inscribed $\odot$ s.



$$\text{To prove} \quad \frac{P}{P'} = \frac{OA}{O'A'} = \frac{OH}{O'H'}.$$

$$\text{Proof.} \quad \frac{P}{P'} = \frac{AB}{A'B'}. \quad (415)$$

In the isosceles $\triangle$ s OAB , $O'A'B'$,

$$\angle AOB = \angle A'O'B'. \quad (412)$$

$\therefore$ the isosceles $\triangle$ s OAB , $O'A'B'$ are similar, (309)

and $\therefore$ the rt. $\triangle$ s $OA H$, $O'A' H'$ are similar, (309)

$$\therefore \frac{AB}{A'B'} = \frac{OA}{O'A'}, \text{ and } \frac{OA}{O'A'} = \frac{OH}{O'H'}. \quad (307)$$

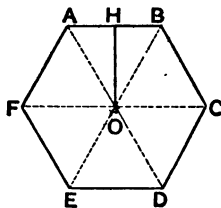
$$\therefore \frac{P}{P'} = \frac{OA}{O'A'} = \frac{OH}{O'H'}. \quad (\text{Ax. 1})$$

Q. E. D.

418. COR. *The areas of regular polygons of the same number of sides are to each other as the squares of the radii of their circumscribed, or of their inscribed circles.* (380)

Proposition 5. Theorem.

419. *The area of a regular polygon is equal to half the product of its perimeter and apothem.*



Hyp. Let P denote the perimeter and R the apothem OH , of the regular polygon $ABCDEF$.

To prove $\text{area } ABCDEF = \frac{1}{2} P \times R$.

Proof. Join OA , OB , OC , etc.

The polygon is divided into equal Δ s whose bases are the equal sides of the polygon, and whose common altitude is the apothem.

Then, $\text{area } OAB = \frac{1}{2} AB \times OH$,
the area of a $\Delta = \frac{1}{2}$ the product of its base and altitude (366).

Similarly, $\text{area } OBC = \frac{1}{2} BC \times OH$,

and so on for all the Δ s of the polygon.

$\therefore \text{area } OAB + \text{area } OBC + \text{etc.} = \frac{1}{2}(AB + BC + \text{etc.}) OH$.

$\therefore \text{area of the polygon } ABCDEF = \frac{1}{2} P \times R$. Q.E.D.

420. COR. *The area of any polygon that circumscribes a circle is equal to half the product of its perimeter and the radius of the circle.*

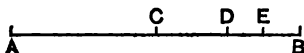
THE MEASURE OF THE CIRCLE.

421. DEF. A *variable quantity*, or simply a *variable*, is a quantity which may have an indefinite number of different successive values.

A *constant* is a quantity whose value does not change in the same discussion.

422. Limit. When the successive values of a variable, under the conditions imposed upon it, approach more and more nearly to the value of some constant quantity, which it can never equal, yet from which it may be made to differ by as small as we please, the constant is called the *limit* of the variable; the variable is said to *approach indefinitely to its limit*.

423. For example, suppose a point to start at A and move along AB towards B under the



condition that, during successive seconds of time, the point moves first half the distance from A to B, that is to C; then half the remaining distance, or to D, then half the remaining distance, or to E, and so on indefinitely. Then the distance from A to the moving point is a *variable* whose limit is the distance AB. For, however long the point may move, under these conditions, there will always remain *some* distance between it and the point B, so that the distance from A to the moving point can never equal AB; but as the moving point can be brought as near as we please to B, its distance from A can be made to differ from the distance AB by as little as we please.

If we call the distance AB 2, then the distance the point moves the first second will be 1, the distance moved the next second will be $\frac{1}{2}$, the distance moved the third second will be $\frac{1}{4}$, and so on. Therefore, the whole distance from A

to the moving point at the end of n seconds is the sum of n terms of the series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \text{etc.}$$

Now it is evident that, however many terms of this series are taken, the sum can never equal 2; but by taking a great number of terms the sum can be made to differ from 2 by as little as we please. Hence we say that the *limit* of the sum of the series as the number of terms is indefinitely increased is 2.

The limit of the fraction $\frac{1}{x}$, as x is indefinitely increased, is zero; for, by increasing x at pleasure, $\frac{1}{x}$ may be made to approach as near as we please to the value zero, but can never be made exactly equal to zero.

424. Principle of Limits. THEOREM. *If two variables are always equal and each approaches a limit, the two limits must be equal.*

For, two variables that are always equal have always the same common value, that is, they are really but a single variable; and it is clear that a single variable cannot at the same time approach indefinitely to two unequal limits.

425. COR. *If two variables, while approaching their respective limits, are always in the same ratio, their limits are in the same ratio.*

For, if x and y are two variables in the constant ratio m , so that $x = my$, then the two variables x and my are always equal to each other, and their limits are equal (424); therefore, if a and b are the limits of x and y respectively, we have $a = mb$; that is, a and b are in the same constant ratio m .

EXERCISE.

The apothem of a regular pentagon is 3 and a side is 2; find the perimeter and area of a regular pentagon whose apothem is 6.

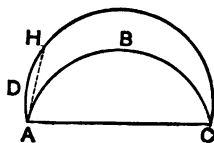
Proposition 6. Theorem.

426. *Every convex curve is less than any enveloping line whatever that has the same extremities.*

Hyp. Let ABC be a convex curve, and $ADHC$ any line enveloping it and terminating at A and C .

To prove that $ABC < ADHC$.

Proof. Of all the lines enveloping the area ABC , there must be at least one line shorter than any other.



Now $ADHC$ cannot be this line.

For, draw the tangent AH to the curve ABC .

Then the line $AHC <$ the line $ADHC$,

since the st. line AH is $<$ the curve ADH (Ax. 10).

$\therefore ADHC$ is not the shortest line.

In the same way it may be shown that no other enveloping line can be the shortest.

$\therefore$ the curve ABC is less than any enveloping line. Q.E.D.

427. COR. 1. *The circumference of a circle is less than the perimeter of any circumscribed polygon, and greater than the perimeter of any inscribed polygon.*

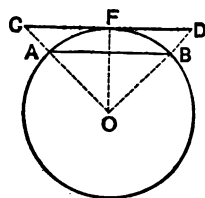
428. COR. 2. *The perimeter of a regular inscribed polygon is less than the perimeter of a regular inscribed polygon of double the number of sides.*

429. COR. 3. *The perimeter of a regular circumscribed polygon is greater than the perimeter of a regular circumscribed polygon of double the number of sides.*

Proposition 7. Theorem.

430. *If a regular polygon be inscribed in a given circle, and another regular polygon be circumscribed about the same circle, and if the number of sides of the polygons be increased indefinitely, then the perimeter of each of the polygons approaches the circumference of the given circle as its limit, and the area of each approaches the area of the circle as its limit.*

Hyp. Let O be the centre of the given $\odot$, and AB, CD homologous sides of the regular inscribed and circumscribed polygons of the same number of sides; let p and P denote the perimeters, s and S the areas of the inscribed and circumscribed polygons respectively.



To prove that the limit of p and P is the $\odot$ ce of the $\odot$,
and the limit of s and S is the area of the $\odot$.

Proof. Join OF and OC .

Then, since the polygons are similar,

$$\therefore P : p = OC : OF, \quad (417)$$

$$\text{and} \quad S : s = \overline{OC}^2 : \overline{OF}^2. \quad (418)$$

Now let the number of sides be increased indefinitely, always remaining the same in each polygon; then the angle FOC will diminish indefinitely, and the pt. C will approach the pt. F , as near as we please.

$\therefore OC$ will approach OF as its limit,

and $\overline{OC}^2$ will approach $\overline{OF}^2$ as its limit.

$\therefore$ ultimately P and p , and also S and s , will differ from each other by less than any assignable quantity.

$\therefore$ the $\odot$ ce of the $\odot$, which is always intermediate between P and p (427), is the limit of either of the perimeters;

and the area of the $\odot$, which is clearly always intermediate between S and s , is the limit of either of the areas.

Q.E.D.

431. COR. *When the number of sides of the polygons is increased indefinitely, the radius of the circle is the limit of the radius of the circumscribed polygon and the apothem of the inscribed polygon.*

432. DEF. Similar arcs, sectors, and segments are those which correspond to equal angles at the centre, in circles of different radii.

EXERCISES.

1. The apothem of a regular pentagon is 6 and a side is 4: find the perimeter and area of a regular pentagon whose apothem is 8.

2. If a point be taken within a regular polygon, prove that the sum of its distances from the sides of the polygon is equal to the radius of the inscribed circle multiplied by the number of sides of the polygon.

3. ABCDE is a regular pentagon; AC and BE are joined so as to intersect in F: prove that CDEF is a parallelogram.

4. The radius of a circle is 8: find the apothem, perimeter, and area of the inscribed equilateral triangle.

5. The radius of a circle is 10: find the perimeter and area of the regular inscribed octagon.

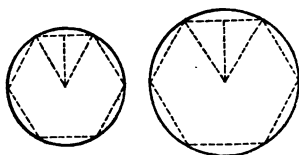
6. The radius of a circle is 4: find the area of the inscribed square.

Proposition 8. Theorem.

433. *The circumferences of two circles are to each other as their radii, and the areas of two circles are to each other as the squares of their radii.*

Hyp. Let C and C' be the $\odot$ es, R and R' the radii, and S and S' the areas of the two $\odot$ s.

To prove $C : C' = R : R'$,
and $S : S' = R^2 : R'^2$.



Proof. Inscribe in the $\odot$ s two regular polygons of the same number of sides.

Let P and P' be their perimeters, and A and A' their areas.

Then, because the polygons are regular, with the same number of sides,

$$P : P' = R : R', \quad (417)$$

$$\text{and} \quad A : A' = R^2 : R'^2. \quad (418)$$

Now let the number of sides of each polygon be increased indefinitely, the number remaining always equal in each.

Then P and P' will approach C and C' as their limits, (431)

and A and A' will approach S and S' as their limits. (431)

Since the above ratios remain the same whatever be the number of sides in the polygons, so long as the number is the same in each (417), their limits are in the same ratios.

(425)

$$\therefore C : C' = R : R', \text{ and } S : S' = R^2 : R'^2.$$

Q. E. D.

$$\mathbf{434. COR. 1.} \quad \frac{C}{C'} = \frac{2R}{2R'}, \quad \text{and} \quad \frac{S}{S'} = \frac{4R^2}{4R'^2}. \quad (294)$$

Therefore, *the circumferences of circles are to each other as their diameters, and their areas are to each other as the squares of their diameters.*

435. COR. 2. Since similar arcs and sectors are like parts of the respective $\odot$ ces and $\odot$ s to which they belong (432), therefore :

Similar arcs are to each other as their radii, and similar sectors are to each other as the squares of their radii.

436. COR. 3. By alternation, we have, from (434),

$$C : 2 R = C' : 2 R'.$$

That is, the ratio of one $\odot$ ce to its diameter is the same as the ratio of any other $\odot$ ce to its diameter. Therefore :

The ratio of the circumference of a circle to its diameter is constant.

This constant ratio is usually represented by the Greek letter π , so that

$$\frac{C}{2R} = \pi; \therefore C = 2\pi R.$$

437. SCH. The ratio π is incommensurable, and can therefore be expressed only approximately in numbers, though the letter π is used to *represent* its *exact* value.

We give here the value of π , its reciprocal, and its logarithm:

$$\pi = 3.14159\ 26535\ 89793\ 23846\ \dots\dots$$

$$\frac{1}{\pi} = 0.31830\ 98861\ 83790\ 67153\ \dots\dots$$

$$\log \pi = 0.49714\ 98726\ 94133\ 85435\ \dots\dots$$

Proposition 9. Theorem.

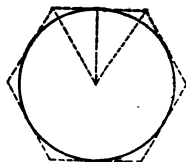
438. *The area of a circle is equal to half the product of its radius and circumference.*

Hyp. Let S be the area, R the radius, and C the $\odot$ ce of the $\odot$.

To prove

$$S = \frac{1}{2} R \times C.$$

Proof. Circumscribe about the $\odot$ any regular polygon; let P be its perimeter, and A its area.



Then

$$A = \frac{1}{2} P \times R. \quad (419)$$

Now let the number of sides of the polygon be increased indefinitely.

Then the perimeter of the polygon will approach the $\odot$ ce of the $\odot$ as its limit, (431)

and the area of the polygon will approach the area of the $\odot$ as its limit. (431)

But the above equality remains true whatever be the number of sides of the polygon.

$\therefore$ in the limit we have

$$S = \frac{1}{2} R \times C. \quad (425)$$

Q.E.D.

439. Cor. 1. By (436), $C = 2\pi R$.

Substituting above, we obtain

$$S = \pi R^2.$$

Therefore, *the area of a circle is equal to the square of the radius multiplied by the constant ratio π .*

440. Cor. 2. Let s denote the area of a sector, and c the arc.

Then, since the sector is the same part of the $\odot$ that the arc is of the $\odot$ ce,

$$\therefore s : S = c : C. \quad (294)$$

$$\therefore s = \frac{c \times S}{C} = \frac{c \times \frac{1}{2}R \times C}{C}, \quad (438)$$

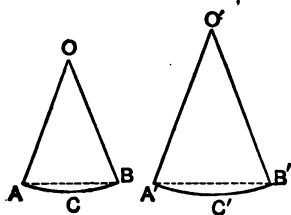
or $s = \frac{1}{2}R \times c.$

Therefore, the area of a sector is equal to half the product of its radius and arc.

Proposition 10. Theorem.

441. The areas of two similar segments are to each other as the squares of their radii.

Hyp. Let S and S' be the areas of the two similar sectors AOB , $A'O'B'$, T and T' the areas of the two similar $\triangle$ s AOB , $A'O'B'$, R and R' the radii.



To prove $\frac{S - T}{S' - T'} = \frac{R^2}{R'^2}.$

Proof. Since $\angle O = \angle O',$ (432)

$$\therefore S : S' = R^2 : R'^2, \quad (435)$$

and $T : T' = R^2 : R'^2. \quad (377)$

$$\therefore \frac{S}{S'} = \frac{T}{T'} = \frac{R^2}{R'^2}.$$

$$\therefore \frac{S - T}{S' - T'} = \frac{R^2}{R'^2} \text{ (by division).} \quad \text{Q.E.D.}$$

EXERCISES.

1. What is the area of a circle whose radius is 40 feet?
2. What is the diameter of a circle whose circumference is 57 yards?

PROBLEMS OF CONSTRUCTION.

Proposition 11. Problem.

442. To inscribe a square in a given circle.

Given, the $\odot$ ABCD, with centre O.

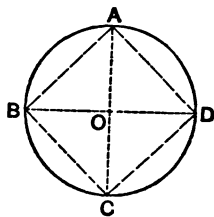
Required, to inscribe a square in it.

Cons. Draw the diameters AC, BD $\perp$ to each other.

Join AB, BC, CD, DA.

Then ABCD is the square required.

Proof. Since the $\angle$ s at O are all rt $\angle$ s, (Cons.)



$\therefore$ the sides AB, BC, etc., are equal,

in the same $\odot$ equal $\angle$ s at the centre intercept equal chords (196),

and the $\angle$ s BAD, ADC, etc., are rt. $\angle$ s,

every $\angle$ inscribed in a $\frac{1}{2}\odot$ is a rt. $\angle$ (240).

$\therefore$ the figure ABCD is a square. (126)

Q.E.F.

443. COR. 1. If tangents be drawn to the circle at the points A, B, C, D, the figure so formed will be a circumscribed square.

444. COR. 2. To inscribe, and circumscribe, regular polygons of 8 sides, bisect the arcs AB, BC, CD, DA, and proceed as before.

By repeating this process, regular inscribed and circumscribed polygons of 16, 32, . . . , and, in general, of $2n$ sides, may be drawn.

EXERCISES.

1. What is the area of a square inscribed in a circle whose area is 48 feet?

2. What is the area of a regular hexagon inscribed in a circle whose area is 560 square feet?

Proposition 12. Problem.

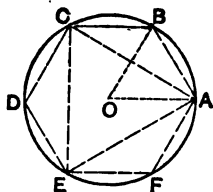
445. *To inscribe a regular hexagon in a given circle.*

Given, the $\odot ABC$, with centre O .

Required, to inscribe a regular hexagon in it.

Cons. With any pt. A on the $\odot$ as a centre, and AO as a radius, describe an arc cutting the $\odot$ in B .

Join AB , BO , OA .



Then AB is a side of the hexagon required.

Proof. Since the $\triangle OAB$ is equilateral, (Cons.)

$\therefore$ it is equiangular. (113)

$\therefore \angle AOB$ is $\frac{1}{3}$ of 2 rt. $\angle$ s, or $\frac{1}{3}$ of 4 rt. $\angle$ s. (103)

$\therefore$ the arc AB is $\frac{1}{3}$ of the $\odot$.

$\therefore$ the chord AB is a side of the regular inscribed hexagon. (400)

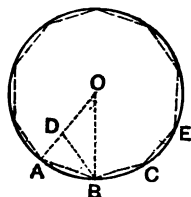
$\therefore$ the figure $ABCDEF$, completed by drawing the chords BC , CD , DE , EF , FA , each equal to the radius OA , is the regular inscribed hexagon required. Q.E.F.

446. COR. 1. *If the alternate vertices of the regular hexagon be joined, we obtain the inscribed equilateral triangle ACE .*

447. COR. 2. *If tangents be drawn to the circle at the points A , B , C , D , E , F , the figure so formed will be a regular circumscribed hexagon.*

448. COR. 3. *To inscribe, and circumscribe, regular polygons of 12 sides, bisect the arcs AB , BC , CD , etc., and proceed as before.*

By repeating this process, regular inscribed and circumscribed polygons of 24, 48, etc., sides may be drawn.

Proposition 13. Problem.**449.** To inscribe a regular decagon in a given circle.*Given*, the $\odot$ ABCE.*Required*, to inscribe in it a regular decagon.*Cons.* Draw any radius OA, and divide it in extreme and mean ratio at D, so that

$$OA : OD = OD : AD. \quad (352)$$

With centre A and OD as a radius, describe an arc cutting the $\odot$ ce at B, and join AB.

Then AB is a side of the required inscribed decagon.

Proof. Join BO, BD.Since $OA : OD = OD : AD$, (Cons.)and $OD = BA$, (Cons.)

$$\therefore OA : BA = BA : AD.$$

 $\therefore$ the $\triangle$ s OAB and BAD are similar,*having $\angle A$ common and the including sides proportional* (814).

$$\therefore \angle ABD = \angle AOB. \quad (307)$$

Because

 $OA = OB$, and $BA = OD$, (Radii) and (Cons.)

$$\therefore OB : BA = OD : AD,$$

$$\therefore \angle ABD = \angle DBO. \quad (303)$$

$$\therefore \angle ABO = 2 \angle ABD = 2 \angle AOB.$$

$$\therefore \text{also } \angle BAO = \angle ABO = 2 \angle AOB,$$

being opp. the equal sides of an isosceles $\triangle$ (111).

$$\therefore \angle ABO + \angle BAO + \angle AOB = 5 \angle AOB = 2 \text{ rt. } \angle \text{s.}$$

$$\therefore \angle AOB = \frac{1}{5} \text{ of } 2 \text{ rt. } \angle \text{s.}, \text{ or } \frac{1}{10} \text{ of } 4 \text{ rt. } \angle \text{s.}$$

$$\therefore \text{the arc AB is } \frac{1}{10} \text{ of the } \odot \text{ce.}$$

 $\therefore$ the chord AB is a side of the regular inscribed decagon. (400) $\therefore$ the figure ABCE . . . , formed by applying AB ten times to the $\odot$ ce, is the regular inscribed decagon required.

Q. E. F

450. COR. 1. *If the alternate vertices of the required decagon be joined, a regular pentagon is inscribed.*

451. COR. 2. *If tangents be drawn at the points at which the circumference is divided, the figure so formed will be a regular circumscribed decagon.*

452. COR. 3. *To inscribe, and circumscribe, regular polygons of 20 sides, bisect the arcs AB , BC , etc., and proceed as before.*

By repeating this process, regular inscribed and circumscribed polygons of 40, 80, etc., sides may be drawn.

EXERCISES.

1. The side of an inscribed square is equal to the radius of the circle multiplied by $\sqrt{2}$.

2. The side of an inscribed equilateral triangle is equal to the radius multiplied by $\sqrt{3}$.

3. The apothem of an inscribed square is equal to half the radius multiplied by $\sqrt{2}$.

4. The apothem of an inscribed equilateral triangle is equal to half the radius.

5. The apothem of a regular inscribed hexagon is equal to half the radius multiplied by $\sqrt{3}$.

6. The area of a circumscribed square is double the area of the inscribed square.

7. Required the area of an equilateral triangle inscribed in a circle whose radius is 4.

8. Required the area of a square inscribed in a circle whose radius is 5.

9. Required the area of a regular hexagon inscribed in a circle whose radius is 8.

10. Required the area of a circle whose circumference is 100.

11. Required the area of a circle inscribed in a square whose area is 36.

Proposition. 14. Problem.

453. *To inscribe a regular pentadecagon in a given circle.*

Given, the $\odot$ ABC.

Required, to inscribe in it a regular polygon of 15 sides.

Cons. Lay off HB equal to a side of a regular inscribed hexagon. (445)

Lay off HA equal to a side of a regular inscribed decagon. (449)

Join AB.

Then AB is a side of the required inscribed pentadecagon.

Proof. Arc AB = arc HB - arc HA

$$= \frac{1}{6} \text{ of } \odot ce - \frac{1}{10} \text{ of } \odot ce = \frac{1}{15} \text{ of } \odot ce.$$

$\therefore$ the chord AB is a side of the regular inscribed pentadecagon. (400)

$\therefore$ the figure ABCD . . . , formed by applying AB 15 times to the $\odot ce$, is the regular inscribed pentadecagon required. Q.E.F.

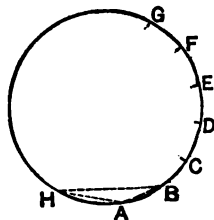
454. COR. 1. *If tangents be drawn at the points at which the circumference is divided, a regular circumscribed pentadecagon is obtained.*

455. COR. 2. *To inscribe, and circumscribe, regular polygons of 30 sides, bisect the arcs AB, BC, etc., and proceed as before.*

By repeating this process, regular inscribed and circumscribed polygons of 60, 120, etc., sides may be drawn.

456. SCH. These are the only polygons that could be constructed by the ancient geometers, by the use of the rule and compass. About the beginning of the present century, Gauss, the great German mathematician, proved that whenever $2^n + 1$ is a prime number, and n an integer, a regular polygon of that number of sides could be inscribed in the circle, by the rule and compass.

Therefore, it is possible to inscribe regular polygons of

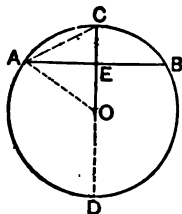


Proposition 16. Problem.

459. *Given the radius and the side of a regular inscribed polygon, to compute the side of the regular inscribed polygon of double the number of sides.*

Given, AB, a side of the regular inscribed polygon, and $OC = R$, the radius of the $\odot ABD$.

Required, to compute AC, a side of the regular inscribed polygon of double the number of sides.



Cons. C is the mid. pt. of the arc AB. (403)

Produce CO to D, and join OA. (204)

CD is $\perp$ to AB at its mid. pt. (325)

AC is a mean proportional between CD and CE. (325)

$$\therefore \overline{AC}^2 = CD \times CE = R(2R - 2OE).$$

But $OE = \frac{1}{2} \sqrt{4R^2 - \overline{AB}^2}.$ (457)

$$\therefore AC = \sqrt{R(2R - \sqrt{4R^2 - \overline{AB}^2})}. \quad \text{Q.E.F.}$$

460. SCH. When $R = 1$, this gives

$$AC = \sqrt{2 - \sqrt{4 - \overline{AB}^2}}.$$

By repeated applications of this formula, we may compute successively the sides, and therefore the perimeters, of the regular inscribed polygons of $2n, 4n, 8n, 16n, \dots$ sides, n being the number of sides of the first polygon.

EXERCISES.

1. If the radius of a circle is 4, find its circumference and area.

2. If the circumference of a circle is 28, find its diameter and area.

Proposition 17. Problem.

461. To compute the ratio of the circumference of a circle to its diameter.

Given, the $\odot$ ce C, and the radius R.

Required, to find the number π .

Cons. $C = 2\pi R$. (436)

Let $R = 1$, then $\pi = \frac{1}{2}C$.

That is, the number $\pi = \text{semi } \odot\text{ce of radius } 1$.

Therefore, the semi-perimeter of each polygon inscribed in this $\odot$ ce is an approximate value of π : and as the number of sides of the polygons increases indefinitely, the lengths of the perimeters approach to that of the $\odot$ ce as the limit. (430)

Hence, by the process of (460), we may obtain a succession of nearer and nearer approximations to the length of the semi $\odot$ ce.

It is convenient to begin the computation with the inscribed hexagon.

$\therefore$ making $AB = 1$, we have, from (460), the following:

Number of Sides.	Semi-perimeters.	Number of sides.	Semi-perimeters.
6	3.00000000	96	3.14103198
12	3.10582854	192	3.14145255
24	3.13262861	384	3.14155772
48	3.13935026	768	3.14158471

The last two results show that the first four decimals do not change as the number of sides is increased.

Hence the approximate value of π is 3.1415, correct to the fourth decimal place. Q.E.F.

In practice we generally take $\pi = 3.14159$.

462. SCH. The above is called the *method of perimeters*. For the *method of isoperimeters* see Rouché et Comberousse, Edition of 1883, p. 194.

NOTE.—The number π is of such fundamental importance in geometry that mathematicians have sought for its value in a great variety of ways, all of which agree in the conclusion that it cannot be expressed exactly in decimals, but only approximately. Archimedes (born 287 B.C.) was the first to assign an approximate value of π . He proved that it is included between the numbers $3\frac{1}{4}$ and $3\frac{1}{2}$, or, in decimals, between 3.1428 and 3.1408; he therefore assigned its value correctly within a unit of the third decimal place. The number $3\frac{1}{4}$, or $\frac{13}{4}$ is often used in rough computations.

Adrian Metius, a Dutch geometer of the 16th century, has given us the much more accurate value of $\frac{10}{33}$, which is correct to within a half-millionth, and which is remarkable for the manner in which it is formed by the first three odd numbers 1, 3, 5, each written twice.

More recently, the value has been found to a great number of decimals by the aid of series. Dase and Clausen, German computers, carried the calculation to 200 decimal places, independently of each other, and their results agreed to the last figure.

The first 20 figures of their result are as follows :

$$3.14159\ 26535\ 89793\ 23846. \quad (437)$$

This result is far beyond all the wants of mathematics. Ten decimals are sufficient to give the circumference of the earth to the fraction of an inch, and thirty decimals would give the circumference of the whole visible universe to a quantity imperceptible with the most powerful microscope.*

EXERCISES.

1. The area of the regular inscribed hexagon is equal to twice the area of the inscribed equilateral triangle.
2. The square of a side of the inscribed equilateral triangle is three times the square of a side of the regular inscribed hexagon.
3. The area of a regular inscribed hexagon is half the area of the circumscribed equilateral triangle.
4. If the diameter of a circle be produced to C until the produced part is equal to the radius, the two tangents from C and their chord of contact form an equilateral triangle.

* Newcomb's Geom., p. 235.

5. Divide an angle of an equilateral triangle into five equal parts.

Describe a $\odot$ about the Δ , then use (453).

6. The square inscribed in a semicircle is equal to two-fifths the square inscribed in the whole circle.

7. The area of a given circle is 314.16; if this circle be circumscribed by a square, find the area of the part between the circumference and the perimeter of the square.

8. The area of a circle is 40 feet; find the side of the inscribed square.

9. Find the angle subtended at the centre of a circle by an arc 6 feet long, if the radius is 8 feet long.

10. Find the length of the arc subtended by one side of a regular octagon inscribed in a circle whose radius is 20 feet.

11. Find the area of a circular sector whose arc contains 18° , the radius of the circle being 4 feet.

12. Find the area of a circular sector, the chord of half the arc being 20 inches, and the radius 45 inches.

13. The radius of a circle is 5 feet: find the area of a circle 7 times as large.

14. The radius of a circle is 7 feet: find the radius of a circle 16 times as large.

15. Find the height of an arc, the chord of half the arc being 10 feet, and the radius 16 feet.

16. Find the area of a segment whose height is 4 inches, and chord 30 inches.

17. Find the area of a segment whose height is 16 inches, the radius of the circle being 20 inches.

18. Find the area of a segment whose arc is 100° , the radius being 12 feet.

19. Find the area of a sector, the radius of the circle being 24 feet, and the angle at the centre 30° .

20. A circular park 500 feet in diameter has a carriage-way around it 25 feet wide: find the area of the carriage-way.

MAXIMA AND MINIMA.

463. DEF. A *maximum* quantity is the greatest quantity of the same kind; and a *minimum* quantity, is the least quantity of the same kind.

Thus, the diameter of a circle is a maximum among all inscribed straight lines; and a perpendicular is a minimum among all the straight lines drawn from a given point to a given straight line.

464. *Isoperimetric* figures are those which have equal perimeters.

We give here a few simple but important propositions bearing on this part of Geometry.

Proposition 18. Theorem.

465. Of all triangles formed with two given sides, that in which these sides are perpendicular to each other is the maximum.

Hyp. Let ABC , $A'BC$ be two Δ s having the sides AB , BC , respectively equal to $A'B$, BC ; and let $\angle ABC$ be a rt. $\angle$.

To prove $\Delta ABC > \Delta A'BC$.

Proof. Draw $A'D \perp$ to BC .

Since the $\perp$ is the shortest distance from a pt. to a line, (58)

$$\therefore AB > A'D.$$

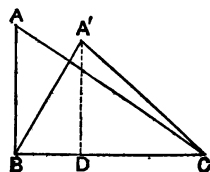
$$\text{But} \quad AB = A'B. \quad (\text{Hyp.})$$

$$\therefore AB > A'D.$$

But AB and $A'D$ are the altitudes respectively of the Δ s ABC , $A'BC$.

$$\therefore \Delta ABC > \Delta A'BC. \quad (369)$$

Q.E.D.



Proposition 19. Theorem.

466. *Of all triangles having equal perimeters and the same base, the isosceles triangle is the maximum.*

Hyp. Let the $\triangle$ s ABC , ABD have equal perimeters and the same base AB , and let the $\triangle ABC$ be isosceles.

To prove $\triangle ABC > \triangle ABD$.

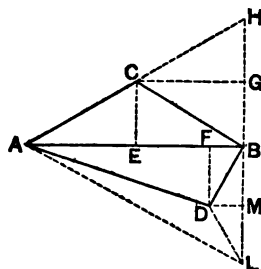
Proof. Produce AC to H , so that

$CH = CB = CA$, and join HB .

Then $\angle ABH$ is a rt. $\angle$,

being inscribed in the $\frac{1}{2}\odot$ whose cent. is C and radius is CB (240).

Produce HB , take $DL = DB$, join AL , and draw CG , $DM \parallel$, and CE , $DF \perp$, to AB .



Then $AD + DL = AD + DB = AC + CB = AH$.

But $AD + DL > AL$, $\therefore AH > AL$.

$$\therefore BH > BL, \quad (65)$$

and $\therefore \frac{1}{2}BH > \frac{1}{2}BL$.

But $\frac{1}{2}BH = BG = CE$, and $\frac{1}{2}BL = BM = DF$. (61)

$$\therefore CE > DF,$$

and $\therefore \triangle ABC > \triangle ABD$. (369)

Q. E. D.

467. COR. *Of all the triangles of the same perimeter, that which is equilateral is the maximum.*

For, the maximum triangle having a given perimeter must be isosceles whichever side is taken as the base.

Proposition 20. Theorem.

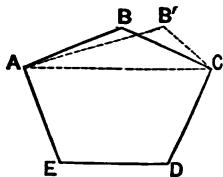
468. *Of all isoperimetric polygons having the same number of sides, the maximum polygon is equilateral.*

Hyp. Let ABCDE be the maximum polygon of given perimeter and given number of sides.

To prove it is equilateral.

Proof. Join AC.

If possible, let the sides AB, BC of the $\triangle ABC$ be unequal, and let $AB'C$ be an isosceles $\triangle$ having the same base AC, and



$$AB' + B'C = AB + BC.$$

$$\text{Then} \quad \triangle AB'C > \triangle ABC. \quad (466)$$

But the area ACDE remains unchanged.

$$\therefore \text{area } AB'CDE > \text{area } ABCDE. \quad (\text{Ax. 4})$$

But area ABCDE is the maximum. (Hyp.)

$$\therefore \text{area } AB'CDE < \text{area } ABCDE,$$

and $\therefore$ AB and BC cannot be unequal.

$$\therefore AB = BC.$$

In the same way it may be proved that

$$BC = CD = DE = \text{etc.} \quad \text{Q.E.D.}$$

EXERCISE.

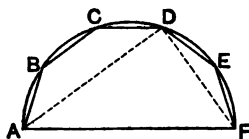
If the diagonals of a quadrilateral are given in magnitude, the area of the quadrilateral is a maximum when the diagonals are at right angles to each other.

Proposition 21. Theorem.

469. *Of all polygons formed of sides all given but one, the maximum can be inscribed in a semicircle with the undetermined side for its diameter.**

Hyp. Let $ABCDEF$ be the maximum polygon formed of the given sides AB, BC, CD, DE, EF .

To prove $ABCDEF$ can be inscribed in a semicircle.



Proof. Join any vertex, as D , with the extremities A and F of the side not given.

Then the $\triangle ADF$ must be the max. of all $\triangle$ s formed with the given sides AD and FD ; for, if it is not, by increasing or decreasing the $\angle ADF$, keeping the sides AD and FD unchanged in length, the $\triangle ADF$ may be increased, while the rest of the polygon, $ABCD, DEF$, remains unchanged; so that the whole polygon $ABCDEF$ is increased.

But this is contrary to the hypothesis that the given polygon is a maximum.

$\therefore$ the $\triangle ADF$ must be the max. of $\triangle$ s formed with the given sides AD and FD .

$\therefore$ the $\angle ADF$ is a rt. $\angle$. (465)

$\therefore D$ is on the semi $\odot$ ce whose diameter is AF . (240)

$\therefore$ any vertex is on the semi $\odot$ ce whose diameter is AF .

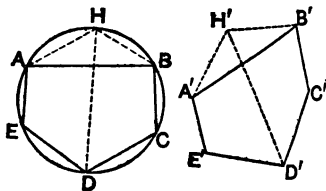
Q. E. D.

* Peirce's Geometry, p. 92.

Proposition 22. Theorem.

470. *Of all polygons formed with the same given sides, that which can be inscribed in a circle is the maximum.*

Hyp. Let $ABCDE$ be a polygon inscribed in a $\odot$, and $A'B'C'D'E'$ any other polygon with the same sides as the first, but which cannot be inscribed in a $\odot$.



To prove

$ABCDE > A'B'C'D'E'$.

Proof. Draw the diameter DH .

Join AH , HB .

Upon $A'B'$ ($= AB$) cons. $\triangle A'B'H' = \triangle ABH$, and join $D'H'$.

Then, since the polygon $HAED$ is inscribed in $\frac{1}{2}\odot$ with diameter HD ,

$$\therefore HAED > H'A'E'D', \quad (469)$$

and

$$HBCD > H'B'C'D'. \quad (469)$$

Adding, $AHBCDE > A'H'B'C'D'E'$.

Subtracting, $\triangle ABH = \triangle A'B'H'$.

$$\therefore ABCDE > A'B'C'D'E'. \quad (\text{Ax. 5})$$

Q. E. D.

471. Cor. *The maximum of all isoperimetric polygons of the same number of sides is regular.*

For, it is equilateral, (468)

and it can be inscribed in a $\odot$, (470)

$\therefore$ the polygon is regular. (400)

EXERCISE.

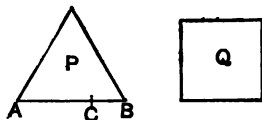
Find a point in a given straight line such that the tangents drawn from it to a given circle contain the greatest angle possible.

Proposition 23. Theorem.

472. *Of regular polygons with a given perimeter, that which has the greatest number of sides has the greatest area.*

Hyp. Let P be a regular polygon of three sides, and Q a regular polygon of four sides, with the same given perimeter.

To prove $Q > P$.



Proof. In any side AB of P take any pt. C .

The polygon P may be regarded as an irregular polygon of four sides, in which the sides AC , CB make with each other a st. $\angle$.

Then, since the polygons P and Q are isoperimetric, and have the same number of sides, (Hyp.)

$\therefore$ the irregular polygon $P <$ the regular polygon Q . (471)

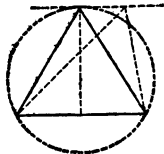
In the same way it may be shown that $Q <$ the regular isoperimetric polygon of five sides, and so on. Q.E.D.

473. COR. *The circle has a greater area than any polygon of the same perimeter.*

EXERCISES.

1. Of all triangles of given base and area, the isosceles is that which has the greatest vertical angle.

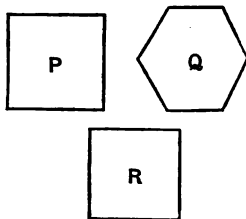
2. The shortest chord which can be drawn through a given point within a circle is the perpendicular to the diameter which passes through that point.



Proposition 24. Theorem.

474. *Of regular polygons having the same given area, the greater the number of sides the less will be the perimeter.*

Hyp. Let P and Q be regular polygons having the same area, and let Q have the greater number of sides.



To prove the perimeter of P > that of Q.

Proof. Let R be a regular polygon having the same perimeter as P and the same number of sides as P.

Then, since Q has the same perimeter as R and a greater number of sides,

(Cons.)

$$\therefore Q > R.$$

(472)

But

$$Q = P,$$

(Hyp.)

$$\therefore P > R.$$

$$\therefore \text{the perimeter of } P > \text{the perimeter of } R. \quad (379)$$

$$\therefore \text{the perimeter of } P > \text{the perimeter of } Q. \quad \text{Q.E.D.}$$

475. COR. *The circumference of a circle is less than the perimeter of any polygon of the same area.*

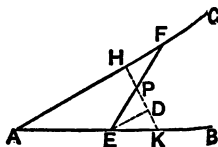
Proposition 25. Theorem.

476. *Given two intersecting straight lines AB, AC, and a point P between them; then of all straight lines which pass through P and are terminated by AB, AC, that which is bisected at P cuts off the triangle of minimum area.*

Hyp. Let EF be the st. line, terminated by AB, AC, which is bisected at P.

To prove $\triangle AEF$ is a minimum.

Proof. Let HK be any other st. line passing through P.



Through E draw ED $\parallel$ to AC.

$$\text{Then} \quad \triangle HPF = \triangle DPE. \quad (105)$$

$$\text{But} \quad \triangle DPE < \triangle KPE.$$

$$\therefore \triangle HPF < \triangle KPE.$$

Add area AEPH to each.

$$\therefore \triangle AEF < \triangle AKH. \quad (\text{Ax. 4})$$

In the same way it may be shown that $\triangle AEF <$ any other $\triangle$ formed by a st. line through P.

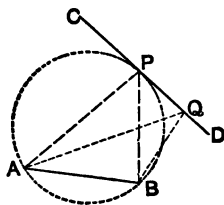
$$\therefore \triangle AEF \text{ is a minimum.} \quad \text{Q.E.D.}$$

Proposition 26. Problem.

477. *To find at what point in a given straight line the angle subtended by the line joining two given points, which are on the same side of the given straight line, is a maximum.*

Given, the st. line CD, and the pts. A, B, on the same side of CD.

Required, to find at what pt. in CD the $\angle$ subtended by the st. line AB is a maximum.



Cons. Describe a $\odot$ to pass through A, B, and to touch the st. line CD. (339) and Ex. 40 in (354)

Let P be the pt. of contact.

Then the $\angle APB$ is the required max. $\angle$.

Proof. Take any other pt. in CD as Q, and join AQ, BQ.

$$\text{Then} \quad \angle AQB < \angle APB. \quad (246)$$

$\therefore \angle APB >$ any other $\angle$ subtended by AB at a pt. in CD.

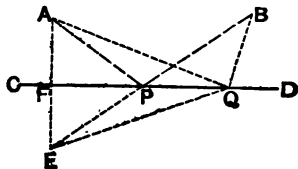
Q.E.F.

Proposition 27. Problem.

478. *In a straight line of indefinite length find a point such that the sum of its distances from two given points, on the same side of the given line, shall be a minimum.*

Given, the st. line CD , and the pts. A, B , on the same side of CD .

Required, to find a pt. P in CD , so that the sum of AP, PB is a minimum.



Cons. Draw $AF \perp$ to CD ; and produce AF to E , making $FE = AF$.

Join EB , cutting CD at P .

Join AP, PB .

Then P is the required pt., and of all lines drawn from A and B to a pt. in CD , the sum of AP, PB is a minimum.

Proof. Let Q be any other pt. in CD .

Join AQ, BQ, EQ .

Then $\text{rt. } \triangle AFP = \text{rt. } \triangle EFP. \quad (104)$

$$\therefore AP = EP.$$

Similarly, $AQ = EQ.$

But $EB < EQ + QB. \quad (96)$

$$\therefore AP + PB < AQ + QB.$$

$\therefore$ the sum of AP and PB is a minimum. Q.E.F.

479. COR. *The sum of AP and PB is a minimum, when these lines are equally inclined to CD ;*

for $\angle APC = \angle EPC = \angle BPD.$

NOTE.—In order that a ray of light from A may be reflected to a point B , it must fall upon a mirror CD at a point P where $\angle APC = \angle BPD$; i.e., by (478) the ray pursues the shortest path between A and B and touching CD .

EXERCISES.

THEOREMS.

1. If AB be a side of an equilateral triangle inscribed in a circle, and AD a side of the inscribed square: prove that three times the square on AD is equal to twice the square on AB .

2. Show that the sum of the perpendiculars from any point inside a regular hexagon to the six sides is equal to three times the diameter of the inscribed circle.

3. The area of the regular inscribed hexagon is twice the area of the inscribed equilateral triangle.

4. The area of the regular inscribed hexagon is three-fourths of that of the regular circumscribed hexagon.

5. The area of the regular inscribed hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

6. If the perpendicular from A to the side BC of the equilateral triangle ABC meet BC in D , and the inscribed circle in G ; prove that $GD = 2AG$.

See figure of (209).

7. If three circles touch each other externally, and a triangle be formed by joining their centres, and another triangle by joining their points of contact, the inscribed circle of the former triangle will be the circumscribed circle of the latter.

8. If $ABCD$ be a square described about a given circle, and P any point on the circumference of the circle; prove that the sum of the squares on PA, PB, PC, PD , is three times the square on the diameter of the circle.

Use (333).

9. ABC is a triangle having each of the angles B, C double the angle A ; the bisectors of the angles B, C meet AC and the circle circumscribing the triangle ABC respectively in D, E : prove that $ADBE$ is a rhombus.

10. $ABCDE$ is a regular pentagon inscribed in a circle, P is the middle point of the arc AB : show that the difference of the straight lines AP, CP is equal to the radius of the circle

11. In the last exercise prove that the sum of the squares on PC and BC is equal to four times the square on the radius.

12. ABCDE is a regular pentagon, BE is drawn cutting AC, AD in F, G respectively: show that the sum of AB and AE is equal to the sum of BE and FG.

Describe a $\odot$ about the pentagon, etc.

13. In a circle a regular pentagon and a regular decagon are inscribed; the middle points of the adjacent sides of the pentagon are joined: prove that the sides of the pentagon so formed are equal to the radius of the circle inscribed in the decagon.

14. Every equilateral polygon inscribed in a circle is equiangular.

15. Every equilateral polygon circumscribed about a circle is equiangular, if the number of sides be odd.

16. Every equiangular polygon inscribed in a circle is equilateral, if the number of sides be odd.

17. Every equiangular polygon circumscribed about a circle is equilateral.

18. The figure formed by the five diagonals of a regular pentagon is another regular pentagon.

19. If the alternate sides of a regular pentagon be produced to meet, the five points of meeting form another regular pentagon.

Let a denote a side of a regular polygon inscribed in a circle whose radius is R ; then:

20. In a regular decagon,

$$a = \frac{R}{2}(\sqrt{5} - 1).$$

Use (449).

21. In a regular pentagon,

$$a = \frac{R}{2}\sqrt{10 - 2\sqrt{5}}. \quad (459 \text{ reciprocally})$$

22. In a regular octagon,

$$a = R\sqrt{2 - \sqrt{2}}. \quad (459)$$

23. In a regular dodecagon,

$$a = R \sqrt{2 - \sqrt{3}}. \quad (459)$$

24. In a regular pentadecagon,

$$a = \frac{R}{4} (\sqrt{10 + 2\sqrt{5}} + \sqrt{3} - \sqrt{15}). \quad (453)$$

25. The side of the regular inscribed pentagon is equal to the hypotenuse of a right triangle whose sides are the radius and the side of the regular inscribed decagon.

NUMERICAL EXERCISES.

26. The diameter of a circle is 5 feet: find the side of the inscribed square. *Ans.* 3.535 ft.

27. The apothem of a regular hexagon is 2: find the area of the circumscribing circle. *Ans.* $5\frac{1}{3} \pi$.

28. There are two gardens; one is a square, and the other is a circle; and they each contain an acre: how much further is it around one than the other?

Ans. 17.268 yards.

29. There are two circles; the diameter of the first is 18 inches, and the area of the second is $2\frac{2}{3}$ times that of the first: what is the diameter of the second?

Ans. 30 inches.

30. The circumference of a circle is 78.54 inches: find (1) its diameter, and (2) its area.

Ans. (1) 25 ins.; (2) 490.875 sq. ins.

31. A circle and a square have each a perimeter of 120 feet: which contains the greater area, and how much?

Ans. The circle, 245.95 sq. ft.

32. What is the width of a ring between two concentric circumferences whose lengths are 160 feet and 80 feet?

Ans. 12.732 feet.

33. Find the side of a square equivalent to a circle whose radius is 40 feet.

34. The radius of a circle is 15 feet: find the radius of a circle just three times as large.

35. The area of a square is 225 square feet: find the area of the inscribed circle.

36. The length of an arc of 180° is πR , where R is the radius of the circle (436): find the length of the arc of $25^\circ 45'$ in the circle whose radius is 9 inches. *Ans.* 4 ins.

37. Find the number of degrees in the arc whose length is equal to the radius of the circle. *Ans.* $57^\circ 17' 44''.8$.

38. Find the number of degrees in the arc whose length is 18 inches, the radius being 5 feet. *Ans.* $17^\circ 11' 19''$.

39. Find the length of the arc of 75° in the circle whose radius is 5 feet. *Ans.* 6.545 feet.

40. Find a side of the circumscribed equilateral triangle, the radius of the circle being R . *Ans.* $2R\sqrt{3}$.

41. Find a side of the circumscribed regular hexagon, the radius of the circle being R . *Ans.* $\frac{2R}{\sqrt{3}}$.

42. Find the length of the arc subtended by one side of a regular dodecagon in a circle whose radius is 12.5 feet. *Ans.* 6.545 feet.

43. Find the area of a sector of 60° in the circle whose radius is 10 inches. *Ans.* 52.3599 sq. ins.

44. The area of a given sector is equal to the square constructed on the radius: find the number of degrees in the arc of the sector. *Ans.* $114^\circ 35' 29''.6$.

45. Find the area of the segment of 60° in the circle whose radius is 2 feet. *Ans.* 0.362344 sq. ft.

46. Find the radius of the circle in which the sector of 45° is .125 square inches. *Ans.* .564 inches.

47. Two tangents make with each other an angle of 60° : required the lengths of the arcs into which their points of contact divide the circle, the radius being 21 inches.

Ans. 44 inches, 88 inches.

48. Venice is due south of Leipsic $5^\circ 55'$: how many miles apart are they, the radius of the earth being 4000 miles? *Ans.* 413 miles.

49. The three sides of a triangle are 9, 10, 17 inches

respectively: find (1) its area, and (2) the area of the inscribed circle.

Ans. (1) 36 sq. ins.; (2) 4π .

See (306), Ex. 3.

50. Mount Washington is visible from a point at sea 87 miles off: required the height of the mountain.

Ans. 6270 feet.

PROBLEMS.

51. To circumscribe a square about a given circle.

52. To inscribe a regular octagon in a given square.

53. Given the base of a triangle, the difference of its sides, and the radius of the inscribed circle: construct the triangle.

From the given base AB, cut off AD = given difference; bisect DB in E, draw $EO \perp$ to AB and = the given rad., etc.

54. Describe a circle which shall touch a given circle and two given straight lines which themselves touch the given circle.

55. In a given circle inscribe a triangle whose angles are as the numbers 2, 5, 8.

See (453).

56. To inscribe a regular hexagon in a given equilateral triangle.

57. To construct a circle equivalent to the sum of two given circles.

58. In a given equilateral triangle, construct three equal circles tangent to each other and to the sides of the triangle, and find the radius of these circles in terms of the side of the triangle.

59. Construct an isosceles triangle having each angle at the base double the third angle.

60. Given the vertical angle of a triangle and the radius of the circumscribed circle: find the locus of the centre of the inscribed circle.

Describe a $\odot$ ABC with the given radius, draw AB cutting off segment ACB containing the given $\angle$; bisect arc AB opp. to C in D, draw any chord DC; bisect $\angle CAB$ by AO meeting CD in O, O is the centre of the inscribed $\odot$: $\angle DOA = \angle OCA + \angle OAC = \angle BAD + \angle OAB = \angle DAO$; $\therefore DO = DA$; $\therefore$ etc.

61. Given a vertex of a triangle, the circumscribed cir-

cle and the centre of the inscribed circle: construct the triangle.

Let A be the vertex, BACD the $\odot$, O the centre of the inscribed $\odot$; join AO, produce it to D; with centre D and radius DO describe the $\odot$ BOC; $\therefore$ BD = DC; $\therefore$ arc BD = arc DC; $\angle$ DBO = $\angle$ DOB = $\angle$ OAB + $\angle$ OBA = $\angle$ DBC + $\angle$ OBA; $\therefore$ $\angle$ OBC = $\angle$ OBA, and $\angle$ OAB = $\angle$ OAC; $\therefore$ etc.

EXERCISES IN MAXIMA AND MINIMA.

62. Two sides of a triangle are given in length: how must they be placed that the area of the triangle may be a maximum?

(465).

63. Given the base and vertical angle of a triangle: construct it so that its area may be a maximum.

64. Of all triangles of given base and area, the isosceles is that which has the least perimeter.

(479).

65. Divide a given straight line so that the rectangle contained by the two segments may be a maximum.

66. A straight rod slips between two straight rulers at right angles to each other: in what position is the rod when the triangle formed by the rulers and the rod is a maximum?

67. Show that the greatest rectangle which can be inscribed in a circle is a square.

68. Of all triangles of given vertical angle and altitude, the isosceles is that which has the least area.

69. Of all rectangles of given area, the square has the least perimeter.

70. Of all polygons having the same number of sides and equal areas, the perimeter of an equilateral polygon is a minimum.

71. A and B are two fixed points without a circle: find a point P on the circumference such that the sum of the squares on AP, BP may be a minimum.

(333).

72. A bridge consists of three arches, whose spans are 49 ft., 32 ft., and 49 ft. respectively: show that the point on either bank of the river at which the middle arch subtends the greatest angle is 63 feet distant from the bridge.

See (477).

73. If the sum of the squares of two lines be given, their sum is a maximum when the lines are equal.

74. Of all triangles having the same base and vertical angle, the isosceles triangle has the sum of the sides a maximum.

75. Of all triangles inscribed in a circle, the equilateral triangle has the maximum perimeter.

SOLID GEOMETRY.

BOOK VI.

PLANES AND SOLID ANGLES.

DEFINITIONS.

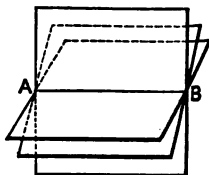
480. A *plane* has been defined (11) as a surface in which the straight line joining any two of its points lies wholly in the surface.

A plane is indefinite in extent, so that, however far the straight line is produced, all its points lie in the plane. But to represent a plane in a diagram, it is necessary to take a limited portion of it, and it is usually represented by a parallelogram supposed to lie in the plane.

481. A plane is said to be *determined* by certain lines or points, when it is the only plane which contains these lines or points.

482. Any number of planes may be passed through any given straight line.

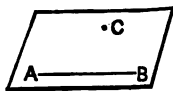
For, if a plane is passed through any given straight line AB, the plane may be turned about AB as an axis, and made to occupy an infinite number of positions, each of which will be a different plane passing through AB.



Hence a given straight line does not determine the position of a plane.

483. A plane is determined by a straight line and a point without that line.

For, if the plane containing the straight line AB, turn about this line as an axis until it contains the given point C, the plane is evidently determined.



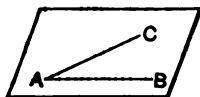
If it is then turned, in either direction, about AB, it will no longer contain the point C.

484. *A plane is determined by three points not in the same straight line.*

For, if we join any two of the points by a straight line, this line and the third point determine a plane. (483)

485. *A plane is determined by two intersecting straight lines.*

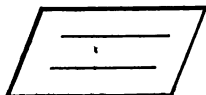
For, a plane passing through AB and any point C in AC, in addition to the point of intersection A, contains the two straight lines AB, AC (480), and is determined.



(483)

486. *A plane is determined by two parallel straight lines.*

For, two parallel straight lines lie in the same plane (68); and since this plane contains either of these parallels and any point in the other, it is determined.



(483)

487. *A straight line is perpendicular to a plane when it is perpendicular to every straight line which it meets in that plane.*

Conversely, the plane in this case is said to be perpendicular to the line.

The point in which a line meets a plane is called the *foot* of the line.

488. *A straight line is parallel to a plane when it never meets the plane, however far both may be produced.*

Conversely, the plane in this case is said to be parallel to the line.

489. *Two planes are parallel when they do not meet, however far they may be produced.*

490. *The projection of a point on a plane is the foot of the perpendicular let fall from the point to the plane.*

491. The *projection of a line on a plane* is the locus of the projections of all the points of this line.

492. The *angle* which a straight line makes with a plane is defined to be the acute angle between the straight line and its projection on the plane, and is called the *inclination* of the line to the plane.

493. By the *distance* of a point from a plane is meant the *shortest* distance from the point to the plane.

494. The line that determines the projection of a point on a plane is called the *projecting line* of that point. The plane including all the projecting lines of a straight line is called the *projecting plane* of the line.

LINES AND PLANES.

Proposition 1. Theorem.

495. If two planes cut each other, their common intersection is a straight line.

Hyp. Let AB, CD be two planes which cut each other.

To prove their common intersection is a straight line.

Proof. Let E and H be any two pts. common to both planes.

Join E and H by the st. line EH.

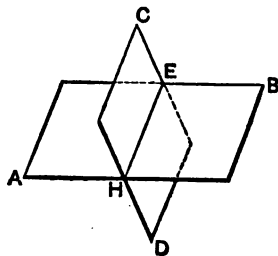
Because E and H are in both planes, the st. line EH lies in both planes. (480)

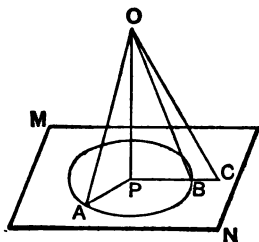
Because a st. line and a pt. out of it cannot lie in two planes, (483)

$\therefore$ EH contains all the pts. common to both planes.

$\therefore$ EH is the common intersection of the two planes.

Q. E. D.



Proposition 2. Theorem.**496.** *If oblique lines are drawn from a point to a plane :**(1) Two oblique lines meeting the plane at equal distances from the foot of the perpendicular are equal.**(2) Of two oblique lines meeting the plane at unequal distances from the foot of the perpendicular, the more remote is the longer.**Hyp.* Let OP be $\perp$ to the plane MN ,and $PA = PB$,and $PC > PA$.*To prove* (1) $OA = OB$,and (2) $OC > OA$.*(1) Proof.* Since OP is common,and $PA = PB$, (Hyp.) $\therefore$ rt. $\triangle OPA =$ rt. $\triangle OPB$. (104) $\therefore OA = OB$.*(2) Proof.* On PC take $PB = PA$, and join OB .Then $OC > OB$. (63)But $OB = OA$. (Just proved) $\therefore OC > OA$. Q.E.D.**497. Cor. 1.** *The perpendicular is the shortest distance from a point to a plane ; therefore, by the distance of a point from a plane is meant the perpendicular distance from the point to the plane.* (493)**498. Cor. 2.** *Equal oblique lines from a point to a plane meet the plane at equal distances from the foot of the perpendicular ; and of two unequal oblique lines, the greater meets the plane at the greater distance from the foot of the perpendicular.***499. Cor. 3.** *The locus of the point in a plane at a given distance from a fixed point without the plane, is a circle whose centre is the foot of the perpendicular.*

Proposition 3. Theorem.

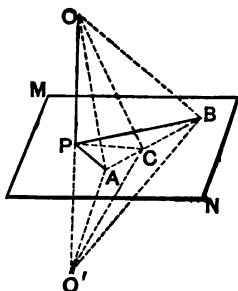
500. *If a straight line is perpendicular to each of two straight lines at their point of intersection, it is perpendicular to the plane of those lines.*

Hyp. Let OP be $\perp$ to PA , PB at the pt. P .

To prove OP is $\perp$ to the plane MN of these lines.

Proof. Join AB , and through P draw in MN any other st. line PC cutting AB in C .

Produce OP to O' making $PO' = PO$, and join O, O' to each of the pts. A, B, C .



Since PA, PB are $\perp$ to OO' at its mid. pt., (*Hyp.*) (*Cons.*)

$$\therefore OA = O'A, \text{ and } OB = O'B. \quad (66)$$

$$\therefore \triangle OAB = \triangle O'AB, \text{ and } \therefore \angle OAC = \angle O'AC. \quad (108)$$

Also, $\triangle OAC = \triangle O'AC$,

having two sides and the included $\angle$ equal, each to each (104).

$$\therefore OC = O'C, \text{ and } \therefore PC \text{ is } \perp \text{ to } OO' \text{ at its mid. pt. } P. \quad (67)$$

$$\therefore OP \text{ is } \perp \text{ to any st. line in } MN \text{ passing through its foot } P.$$

$$\therefore OP \text{ is } \perp \text{ to the plane } MN. \quad (487)$$

Q. E. D.

501. COR. 1. *At a given point in a plane, only one perpendicular to the plane can be erected.*

For, if there could be two $\perp$ s at the same pt. P , pass a plane through them whose intersection with the plane MN is AP ; then these two $\perp$ s would be both $\perp$ to the line AP at the same pt. P , which is impossible. (51)

502. COR. 2. *From a given point without a plane only one perpendicular can be drawn to the plane.*

For, if OP, OA be two such $\perp$ s, the $\triangle OPA$ contains two rt. $\angle$ s, which is impossible. (101)

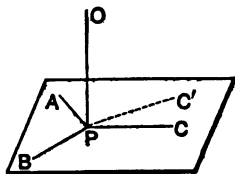
Proposition 4. Theorem.

503. *Conversely, all the perpendiculars to a straight line at the same point lie in a plane perpendicular to the line.*

Hyp. Let PA, PB be two st. lines $\perp$ to OP at P, and PC any other line $\perp$ to OP at P.

To prove PC is in the same plane with PA, PB.

Proof. Let the plane passing through PO and PC cut the plane APB in the line PC'.



Then OP is $\perp$ to PC'. (487)

But in the plane OPC only one $\perp$ can be drawn to OP at P, (51)

$\therefore$ PC and PC' coincide, and PC lies in the plane APB.

Q. E. D.

NOTE.—Hence a plane is determined by one point and the normal to the plane at that point.

504. COR. 1. *At a given point in a straight line one plane, and only one, can be drawn perpendicular to the line.*

505. COR. 2. *If a right angle be turned round one of its arms as an axis, the other arm will generate a plane.*

506. COR. 3. *Through a given point without a straight line one plane, and only one, can be drawn perpendicular to the line.*

For, in the plane of OP and the pt. C, the $\perp$ CP can be drawn to OP; then the plane generated by turning PC round OP will be $\perp$ to OP; and it is clear that there is only one such $\perp$ plane.

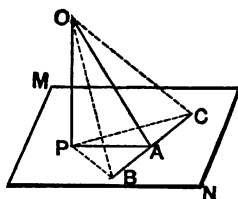
Proposition 5. Theorem.

507. *If from the foot of a perpendicular to a plane a straight line is drawn at right angles to any line in the plane, and its intersection with that line is joined to any point of the perpendicular, this last line will be perpendicular to the line in the plane.*

Hyp. Let OP be a $\perp$ to the plane MN , PA a $\perp$ from P to any line BC in MN , and OA a line joining A with any pt. O in OP .

To prove that OA is $\perp$ to BC .

Proof. Take $AB = AC$, and join PB , PC , OB , OC .



Since PA is $\perp$ to BC at its mid. pt.,

$$\therefore PB = PC, \quad (66)$$

$$\text{and} \quad \therefore OB = OC. \quad (496)$$

Then, since O and A are each equally distant from B and C ,

$$\therefore OA \text{ is } \perp \text{ to } BC. \quad (67)$$

Q. E. D.

508. COR. 1. *The line BC is perpendicular to the plane of the triangle OPA .*

For, it is $\perp$ to the st. lines AP , AO at pt. A . (500)

COR. 2. *The line PA measures the shortest distance between OP and BC .*

EXERCISES.

1. Find the locus of points equally distant from two given points.

2. Given a straight line and any two points: find a point in the straight line equally distant from the two points.

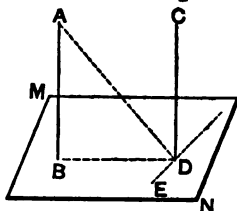
Proposition 6. Theorem.

509. *Two straight lines perpendicular to the same plane are parallel.*

Hyp. Let AB , CD be $\perp$ to the plane MN at the pts. B , D .

To prove $AB \parallel$ to CD .

Proof. Join DB , DA , and draw $DE \perp$ to BD in the plane MN .



Then CD is $\perp$ to ED , (487)

and BD is $\perp$ to ED ; (Cons.)

and AD is $\perp$ to ED . (507)

$\therefore CD$, AD , BD , are all in the same plane,
all the $\perp$ s to a st. line at the same pt. lie in the same plane (508).

$\therefore AB$ and CD lie in the same plane,

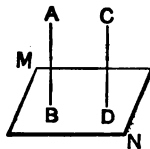
and they are both $\perp$ to BD . (487)

$\therefore AB$ is $\parallel$ to CD . (70)

Q.E.D.

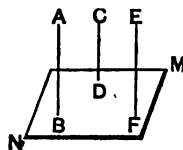
510. COR. 1. *If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to that plane.*

For, if AB is $\parallel$ to CD , and $\perp$ to the plane MN , then a $\perp$ to MN at D will be $\parallel$ to AB (509), and will coincide with CD (501). $\therefore CD$ is $\perp$ to MN .



511. COR. 2. *Two straight lines that are parallel to a third straight line are parallel to each other.*

For, each of the lines AB , CD , is $\perp$ to a plane MN that is $\perp$ to EF (510);
 $\therefore AB$ and CD are $\parallel$. (509)



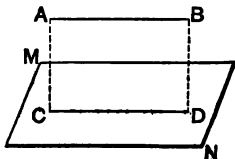
PARALLEL PLANES.

Proposition 7. Theorem.

512. *If two straight lines are parallel, each of them is parallel to every plane passing through the other and not containing both lines.*

Hyp. Let AB , CD be two $\parallel$ st. lines, and MN any plane passing through CD .

To prove $AB \parallel$ to MN .



Proof. The $\parallel$ s AB , CD lie wholly in the plane $ABCD$.

$\therefore$ if AB meets the plane MN , it must meet it in some pt. of the intersection CD , of the two planes.

But AB is $\parallel$ to CD , and so cannot meet it. (Hyp.)

$\therefore AB$ cannot meet the plane MN .

$\therefore AB$ is $\parallel$ to MN .

Q.E.D.

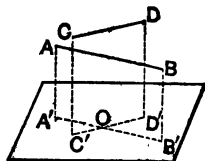
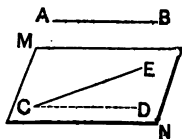
513. COR. 1. *A line parallel to the intersection of two planes is parallel to each of those planes.*

514. COR. 2. *Through any given straight line, a plane can be passed parallel to any other given straight line.*

For, through any pt. C of CE draw $CD \parallel$ to AB ; then the plane of DCE is $\parallel$ to AB . (512)

515. COR. 3. *Through a given point a plane can be passed parallel to any two given straight lines in space.*

For, draw through the given pt. O , in the plane of the given line AB and O , the line $A'B' \parallel$ to AB , and in the plane of the given line CD and O , the line $C'D' \parallel$ to CD ; then the plane of the lines $A'B'$, $C'D'$ is $\parallel$ to each of the lines AB and CD .



(512)

Proposition 8. Theorem.

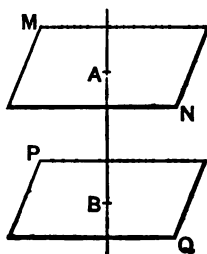
516. *Two planes perpendicular to the same straight line are parallel.*

Hyp. Let the planes MN and PQ be $\perp$ to AB.

To prove MN $\parallel$ to PQ.

Proof. If the planes are not $\parallel$ they will meet if sufficiently produced. (489)

There will then be, through a pt. of their intersection, two planes $\perp$ to the same st. line AB.



But this is impossible.

(506)

$\therefore$ MN is $\parallel$ to PQ.

Q.E.D.

517. COR. 1. *If a straight line is parallel to a plane, the intersection of the plane with any plane passed through the line is parallel to the line.*

Let the student show this. •

518. COR. 2. *If a straight line and a plane are parallel, a parallel to the line drawn through any point of the plane lies in the plane.* (517)

REM. A polygon in space may be formed by joining end to end any number of straight lines, as defined in (137). But in Plane Geometry the lines are all confined to one plane, while there is no such restriction upon polygons in space.

Proposition 9. Theorem.

519. *The intersections of two parallel planes by a third plane are parallel lines.*

Hyp. Let the $\parallel$ planes MN , PQ be cut by the plane AD into the lines AB and CD .

To prove $AB \parallel$ to CD .

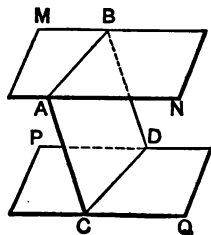
Proof. The lines AB , CD are in the same plane AD .

Because AB lies in the plane MN , and CD in the $\parallel$ plane PQ , and because these planes cannot meet,

$\therefore AB$ and CD cannot meet.

$\therefore AB$ is $\parallel$ to CD .

Q.E.D.



(489)

520. COR. *Parallel straight lines included between parallel planes are equal.*

For, the plane of the $\parallel$ lines AC , BD intersects the $\parallel$ planes MN , PQ in the $\parallel$ lines AB , CD .

$\therefore ABCD$ is a $\square$, and $\therefore AC = BD$. (129)

EXERCISES.

1. Show that all the propositions in Plane Geometry which relate to triangles are true of triangles in space, however situated.

See (485).

2. Show that those propositions are not true of polygons of more than three sides situated in any way in space.

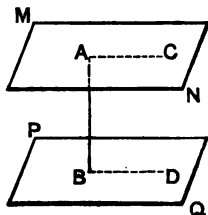
Proposition 10. Theorem.

521. *A straight line perpendicular to one of two parallel planes is perpendicular to the other.*

Hyp. Let MN and PQ be $\parallel$ planes, and let AB be $\perp$ to PQ .

To prove $AB \perp$ to MN .

Proof. Through A draw any st. line AC in the plane MN , and pass a plane through AB , AC , intersecting PQ in BD .



Then AC is $\parallel$ to BD . (519)

But AB is $\perp$ to BD . (487)

$\therefore AB$ is $\perp$ to AC . (71)

And since AC is any line drawn through A in the plane MN ,

$\therefore AB$ is $\perp$ to the plane MN . (487)

Q. E. D.

522. COR. 1. *Through a given point one plane can be passed parallel to a given plane, and only one.*

For, if AB is $\perp$ to PQ , a plane passing through the pt. A , $\perp$ to AB , is $\parallel$ to PQ . (516)

Also, since every plane $\parallel$ to PQ is $\perp$ to AB (521), and since only one plane can be passed through the pt. A $\perp$ to AB (504), therefore only one plane can be passed through a given pt. parallel to a given plane.

523. COR. 2. *Two parallel planes are everywhere equally distant.*

For, all st. lines $\perp$ to the plane PQ are also $\perp$ to the $\parallel$ plane MN (521); and being $\perp$ to the same plane, they are $\parallel$ (509); and being included between $\parallel$ planes, they are equal (520). Hence the planes are everywhere equally distant. (497)

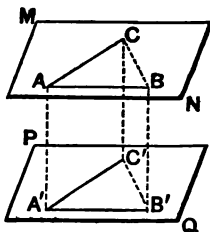
524. COR. 3. *If two intersecting straight lines are each parallel to a given plane, the plane of these lines is parallel to the given plane.*

See (518), (516).

Proposition 11. Theorem.

525. *If two angles not in the same plane have their sides respectively parallel and lying in the same direction, they are equal and their planes are parallel.*

Hyp. Let $\angle$ s A and A' lie in the planes MN , PQ respectively, and let AB be $\parallel$ to $A'B'$ and AC be $\parallel$ to $A'C'$.



(1) *To prove* $\angle A = \angle A'$.

Proof. Take $AB = A'B'$, and $AC = A'C'$, and join AA' , BB' , CC' , BC , $B'C'$.

Then, since AB is = and $\parallel$ to $A'B'$, (Hyp.)

$\therefore AA'$ is = and $\parallel$ to BB' . (133)

In like manner,

AA' is = and $\parallel$ to CC' .

Because BB' and CC' are each = and $\parallel$ to AA' ,

$\therefore BB'$ is = and $\parallel$ to CC' . (511)

$\therefore BC$ is = and $\parallel$ to $B'C'$. (133)

$\therefore \triangle ABC = \triangle A'B'C'$,

having the three sides equal, each to each (108).

$\therefore \angle A = \angle A'$.

(2) *To prove* $MN \parallel PQ$.

Proof. Since the lines AB , AC are each $\parallel$ to the plane PQ , (512)

$\therefore$ their plane MN is $\parallel$ to PQ . (524)

Q. E. D.

EXERCISE.

Find the locus of points equally distant from three given points.

Proposition 12. Theorem.

526. *If two straight lines are cut by three parallel planes, the corresponding segments are proportional.*

Hyp. Let AB, CD be cut by the $\parallel$ planes MN, PQ, RS, in the pts. A, E, B, and C, F, D.

To prove $\frac{AE}{EB} = \frac{CF}{FD}$.

Proof. Join AD cutting the plane PQ in H.

Join EH and FH.

Then, because the $\parallel$ planes PQ, RS are cut by the plane ABD, in the lines EH, BD,

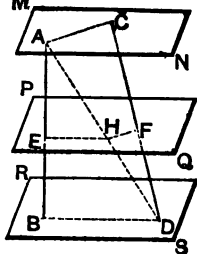
$$\therefore EH \text{ is } \parallel \text{ to } BD. \quad (519)$$

$$\therefore AE : EB = AH : HD. \quad (298)$$

$$\text{In like manner, } HF \text{ is } \parallel \text{ to } AC. \quad (519)$$

$$\therefore AH : HD = CF : FD.$$

$$\therefore AE : EB = CF : FD. \quad \text{Q.E.D.}$$



EXERCISES.

1. To draw a perpendicular to a given plane from a given point without it.

2. To erect a perpendicular to a given plane from a given point in the plane.

3. Prove that through a given line of a given plane, only one plane perpendicular to the given plane can be passed.

DIEDRAL ANGLES.

DEFINITIONS.

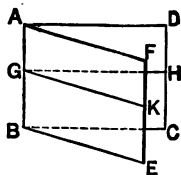
527. When two planes intersect they are said to form with each other a *diedral angle*.

The two planes are called the *faces*, and their line of intersection, the *edge*, of the diedral angle.

Thus, the two planes AC, AE are the faces, and the intersection AB is the edge, of the diedral angle formed by these planes.

A diedral angle is read by the two letters on the edge and one in each face, the two on the edge being read between the other two; or, simply by the two letters on the edge.

Thus, the angle in the figure is read either DABF or AB.



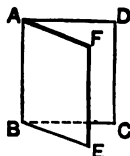
528. If a point is taken in the edge of a diedral angle, and two straight lines are drawn through this point, one in each face, and each perpendicular to the edge, the angle between these lines is called the *plane angle* of the diedral angle. This plane angle is the same at whatever point of the edge it is constructed.

Thus, if at any point G we draw GH and GK in the two faces AC and AE respectively, and both perpendicular to AB, the angle HGK is equal to the angle DAF, since the sides of these angles are respectively parallel. (525)

The plane of the plane angle HGK is perpendicular to the edge AB (500); and conversely, a plane perpendicular to the edge of a diedral angle at any point cuts the faces in lines perpendicular to the edge. (487)

529. A diedral angle may be conceived to be *generated* by revolving a plane about a line of the plane.

Thus, suppose a plane, at first in coincidence with a fixed plane AC , to turn about the edge AB as an axis until it comes into the position AE ; then the magnitude of the diedral angle $DABF$ varies continuously with the *amount* of turning of this plane about AB . The straight line DA , perpendicular to AB , generates the plane angle DAF .

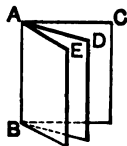


530. Two diedral angles are *equal* when one of them can be applied to the other so that the edges coincide, and the two plane faces of the one coincide respectively with the two plane faces of the other.

The *magnitude* of a diedral angle depends only upon the relative position of its faces, and is independent of their extent.

531. When two diedral angles have a common edge, and the intermediate face common to both, they are said to be *adjacent*.

Thus, the angles $CABD$, $DABE$ are adjacent angles.

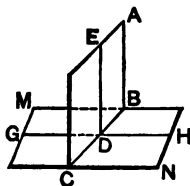


Two diedral angles $CABD$, $DABE$, are added together by placing them adjacent to each other, giving as the sum the diedral angle $CABE$.

532. When the adjacent diedral angles which a plane forms with another plane on opposite sides are equal, each of these angles is called a *right diedral angle*; and the first plane is said to be *perpendicular* to the other.

Thus, if the adjacent diedral angles $ABCM$, $ABCN$ are equal, each of these is a right diedral angle, and the planes AC and MN are perpendicular to each other.

Through a given line in a plane only one plane can be passed perpendicular to the given plane.



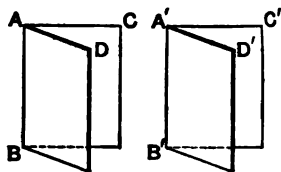
Proposition 13. Theorem.

533. *Two diedral angles are equal if their plane angles are equal.*

Hyp. Let the plane $\angle$ s CAD, C'A'D' of the diedral $\angle$ s AB, A'B' be equal.

To prove

diedral $\angle$ AB = diedral $\angle$ A'B'.



Proof. Apply the diedral $\angle$ AB to the diedral $\angle$ A'B', so that the plane $\angle$ CAD coincides with the equal plane $\angle$ C'A'D'.

Because the pt. A coincides with the pt. A', and the plane CAD with the plane C'A'D', (485)

$\therefore$ AB, the $\perp$ to the plane CAD, coincides with A'B', the $\perp$ to the plane C'A'D'. (501)

Because AB coincides with A'B', and AC with A'C',

$\therefore$ the plane BC coincides with the plane B'C'. (485)

Similarly, the plane BD coincides with the plane B'D'.

$\therefore$ diedral $\angle$ AB = diedral $\angle$ A'B'. (530)

Q. E. D.

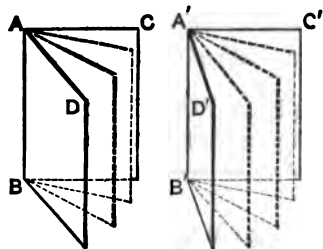
534. SCH. A diedral angle is called *acute*, *right*, or *obtuse* according as the corresponding plane angle is acute, right, or obtuse.

EXERCISE.

Prove that through a line parallel to a given plane, only one plane perpendicular to the given plane can be passed.

Proposition 14. Theorem.

535. *The ratio of any two diedral angles is equal to the ratio of their plane angles.*



Hyp. Let $CABD$, $C'A'B'D'$ be two diedral $\angle$ s, and let CAD , $C'A'D'$ be their plane $\angle$ s.

To prove
$$\frac{AB}{A'B'} = \frac{\angle CAD}{\angle C'A'D'}.$$

Proof. Take any common measure of the $\angle$ s CAD , $C'A'D'$, and suppose it to be contained three times in $\angle CAD$ and 4 times in $\angle C'A'D'$.

Then $\angle CAD : \angle C'A'D' = 3 : 4.$

Pass planes through the edges AB , $A'B'$ and the several lines of division of the plane $\angle$ s CAD , $C'A'D'$ made by the common measure.

These planes divide $CABD$ into 3, and $C'A'B'D'$ into 4 equal parts. (533)

$\therefore CABD : C'A'B'D' = 3 : 4.$

$\therefore CABD : C'A'B'D' = \angle CAD : \angle C'A'D'.$

This proof may be extended to the case where the plane angles are incommensurable, by the method employed in (234), (298), and (356). Q. E. D.

536. SCH. Since the diedral angle and its plane angle increase and decrease in the same ratio, the plane angle is taken as the *measure* of the diedral angle.

See (236).

Proposition 15. Theorem.

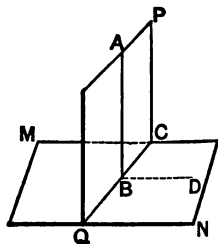
537. *If two planes are perpendicular to each other, a straight line drawn in one of them perpendicular to their intersection is perpendicular to the other.*

Hyp. Let the planes PQ and MN be $\perp$ to each other, and let AB be drawn in PQ $\perp$ to their intersection QC.

To prove AB $\perp$ to MN.

Proof. In the plane MN draw BD $\perp$ to QC at B.

Because BA and BD are each $\perp$ to QC,



(Hyp.) and (Cons.)

$\therefore \angle ABD$ is the plane $\angle$ of the rt. diedral $\angle PCQN$. (528)

Because PQ is $\perp$ to MN, (Hyp.)

$\therefore \angle ABD$ is a rt. $\angle$. (536)

$\therefore$ AB is $\perp$ to QC and BD at their pt. of intersection.

$\therefore$ AB is $\perp$ to the plane MN. (500)

Q. E. D.

538. COR. 1. *If two planes are perpendicular to each other, a straight line through any point of their intersection perpendicular to one of the planes will lie in the other.*

See (501).

539. COR. 2. *If two planes are perpendicular to each other, a straight line from any point of one plane perpendicular to the other will lie in the first plane.*

See (502).

Proposition 16. Theorem.

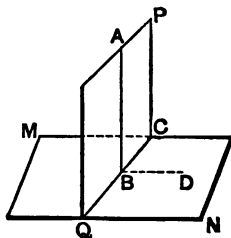
540. *If a straight line is perpendicular to a plane, every plane passed through the line is perpendicular to that plane.*

Hyp. Let AB be $\perp$ to the plane MN , and PQ any plane passed through AB , intersecting MN in QC .

To prove plane $PQ \perp$ to plane MN .

Proof. In the plane MN draw $BD \perp$ to QC at B .

Because AB is $\perp$ to MN , (Hyp.)



$$\therefore AB \text{ is } \perp \text{ to } QC \text{ and } BD. \quad (487)$$

$$\therefore \angle ABD \text{ is the plane } \angle \text{ of the diedral } \angle PCQN. \quad (528)$$

$$\text{But} \quad \angle ABD \text{ is a rt. } \angle. \quad (\text{Proved above})$$

$$\therefore PQ \text{ is } \perp \text{ to } MN. \quad \text{Q.E.D.}$$

541. COR. *A plane perpendicular to the edge of a diedral angle is perpendicular to its faces.*

542. SCH. When three straight lines, AB , CB , DB , are perpendicular to one another at the same point, each line is perpendicular to the plane of the other two, and the three planes are perpendicular to one another.

EXERCISE.

If two planes which intersect contain two lines parallel to each other, the intersection of the planes will be parallel to the lines.

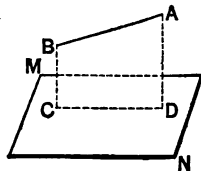
Proposition 17. Theorem.

543. *Through a given straight line oblique to a plane, a plane can be passed perpendicular to the given plane, and but one.*

Hyp. Let AB be the given st. line oblique to the plane MN .

To prove that one plane can be passed through $AB \perp$ to MN , and but one.

Proof. Draw $AD \perp$ to MN from any pt. A of AB .



Through AB and AD pass a plane AC .

Because the plane AC passes through the $\perp AD$, (Cons.)

$\therefore$ the plane AC is $\perp$ to the plane MN . (540)

Also, because any plane passed through $AB \perp$ to MN must contain the $\perp AD$, (539)

$\therefore$ the plane AC is the only plane $\perp$ to MN that can be passed through AB . (485)

Q. E. D.

544. COR. 1. *The projection of a straight line on a plane is a straight line.*

For, the $\perp$ s from all pts. of AB to MN lie in the plane $AC \perp$ to MN (539), and therefore these $\perp$ s all meet MN in the intersection CD of the two planes, which is a straight line. (495)

545. COR. 2. *If a line intersects a plane, its projection passes through the point of intersection.*

Proposition 18. Theorem.

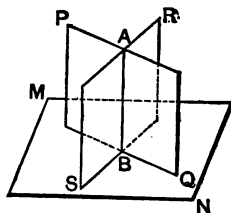
546. *If two intersecting planes are each perpendicular to a third plane, their intersection is perpendicular to the third plane.*

Hyp. Let the planes PQ , RS , intersecting in AB , be $\perp$ to the third plane MN .

To prove $AB \perp$ to MN .

Proof. At the pt. B erect a $\perp$ to the plane MN .

This $\perp$ lies in each of the planes PQ and RS .



(538)

Because this $\perp$ lies in each plane PQ and RS , it is their line of intersection AB .

$\therefore AB$ is $\perp$ to the plane MN .

Q.E.D.

547. COR. 1. *A plane perpendicular to each of two intersecting planes is perpendicular to their intersection.*

548. COR. 2. *If the planes PQ and RS include a right dihedral angle, the three planes PQ , RS , MN , are perpendicular to one another; the intersection of any two of these planes is perpendicular to the third plane; and the three intersections are perpendicular to one another.*

Compare (542).

EXERCISE.

Show that three planes in general intersect in one point: what are the exceptions?

Proposition 19. Theorem.

549. *Every point in the plane bisecting a diedral angle is equally distant from the faces of the angle.*

Hyp. Let AM be the plane which bisects the diedral $\angle CABD$, and let PE , PH be $\perp$ s from any pt. P in the plane AM to the planes CB and DB .

To prove $PE = PH$.

Proof. Let O be the pt. where the plane EPH cuts the line AB .

Join EO , OP , OH .

Because PE is $\perp$ to CB

and PH is $\perp$ to DB ,

(Cons.)

$\therefore$ the plane EPH is $\perp$ to each of the planes CB and DB . (540)

$\therefore$ the plane EPH is $\perp$ to their intersection AB . (547)

$\therefore$ the $\angle$ s POE , POH are the plane $\angle$ s of the diedral $\angle$ s $CABM$ and $DABM$. (528)

Because the diedral $\angle$ s $CABM$, $DABM$ are equal, (Hyp.)

$\therefore \angle POE = \angle POH$. (536)

$\therefore$ rt. $\triangle POE =$ rt. $\triangle POH$,

having the hypotenuse and an acute $\angle =$, each to each (106).

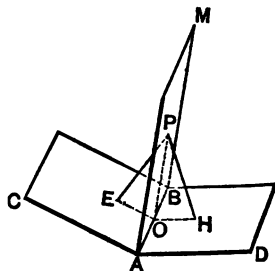
$\therefore PE = PH$.

Q.E.D.

EXERCISES.

1. If three planes have a common line of intersection, the normals drawn to these planes from any point of that line are in one plane.

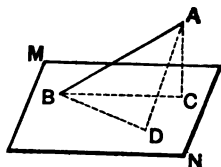
2. If from any point perpendiculars be drawn to the faces of a diedral angle, the angle between these perpendiculars is the supplement of the diedral angle between the planes in which the point is situated.



Proposition 20. Theorem.*

550. *The acute angle between a straight line and its projection on a plane is the least angle which the line makes with any line of the plane.*

Hyp. Let BC be the projection of the st. line AB on the plane MN , and let BD be any other st. line in the plane, drawn through B , the pt. in which AB meets the plane.



To prove $\angle ABC < \angle ABD$.

Proof. Take $BD = BC$, and join AD .

Because in the $\triangle s$ ABC , ABD ,

AB is common, $BD = BC$. (Cons.)

But $AC < AD$. (497)

$\therefore \angle ABC < \angle ABD$. (120)

Q.E.D.

EXERCISES.

1. All points whose projections upon a plane lie in a straight line are themselves in one plane: how is this plane defined?

2. If AB , BC , CD are lines, such that ABC , BCD are right angles, and AB is at right angles to the plane BCD , then will CD be at right angles to the plane ABC .

Proposition 21. Theorem.*

551. *Between two straight lines not in the same plane one, and only one, common perpendicular can be drawn.*

Hyp. Let AB and CD be the given st. lines.

To prove there is one $\perp$, and no more, to both AB and CD .

Proof. Through one of the lines, as CD , pass a plane $MN \parallel$ to the line AB . (514)

Through AB pass the plane $AB' \perp$ to MN , and let their intersection $A'B'$ meet CD at A' .

Then $A'B'$ is $\parallel$ to AB . (517)

At A' erect $A'A$ in the plane $AB' \perp$ to $A'B'$.

$\therefore A'A$ is $\perp$ to the plane MN , (537)

and $\therefore A'A$ is $\perp$ to the line CD . (487)

Because $A'A$ is $\perp$ to $A'B'$, $\therefore$ it is $\perp$ to AB . (71)

$\therefore A'A$ is $\perp$ to both the lines AB and CD .

If there is any other common $\perp$ to AB and CD , let it be HG .

Because HG is $\perp$ to AB ,

$\therefore$ it is $\perp$ to a line HL drawn $\parallel$ to AB in the plane MN , (71)

and $\therefore HG$ is $\perp$ to the plane MN . (500)

But GK in the plane AB' , $\perp$ to $A'B'$, is $\perp$ to MN . (537)

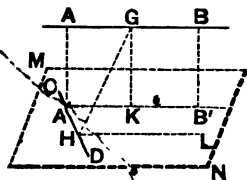
$\therefore$ from the pt. G there are two $\perp$ s GK, GH to MN .

But this is impossible. (502)

$\therefore HG$ is not a common $\perp$ to the lines AB, CD .

$\therefore A'A$ is the only common $\perp$ to AB and CD . Q.E.D.

552. COR. *The perpendicular $A'A$ is the least distance between AB and CD ; for any other line $GH > GK$ (497), or its equal $A'A$.*



EXERCISES.

1. Parallel lines intersecting the same plane make equal angles with it. (492)

2. If a plane bisects a line perpendicularly, every point of the plane is equally distant from the extremities of the line.

3. If three lines in space are parallel, in how many planes may they lie when taken two at a time?

4. If four lines in space are parallel, in how many planes may they lie when taken two at a time?

5. How many different planes may the sides of a quadrilateral in space contain when taken two and two?

6. If two lines not in the same plane are intersected by the same line, how many planes may be determined by the three lines taken two and two?

7. A straight line makes equal angles with parallel planes.

8. The sum of two adjacent diedral angles, formed by one plane meeting another, is equal to two right diedral angles.

9. If two planes intersect each other, the opposite or vertical diedral angles are equal.

10. When a plane intersects two parallel planes, the alternate-interior diedral angles are equal, and the exterior-interior diedral angles are equal.

11. Show that two observations with a spirit-level are sufficient to determine if a plane is horizontal: and prove that for this purpose the two positions of the level must not be parallel.

12. To draw a straight line perpendicular to a given plane from a given point outside of it.

13. To draw a straight line perpendicular to a given plane from a given point in the plane.

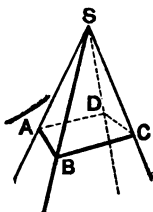
POLYEDRAL ANGLES.

DEFINITIONS.

553. When three or more planes meet in a common point, they are said to form a *polyedral angle* at that point.

The common point in which the planes meet is the *vertex* of the angle, the intersections of the planes are the *edges*, the portions of the planes between the edges are the *faces*, and the plane angles formed by the edges are the *face-angles*.

Thus, the point S is the vertex, the straight lines SA, SB, etc., are the edges, the planes SAB, SBC, etc., are the faces, and the angles ASB, BSC, etc., are the face-angles of the polyedral angle S-ABCD.



554. The edges of a polyedral angle may be produced indefinitely; but to represent the angle clearly, the edges and faces are supposed to be cut off by a plane, as in the figure above. The intersection of the faces with this plane forms a polygon, as ABCD, which is called the *base* of the polyedral angle.

555. In a polyedral angle, each pair of adjacent faces forms a *diedral angle*, and each pair of adjacent edges forms a *face-angle*. There are as many edges as faces, and therefore as many diedral angles as faces.

556. The *magnitude* of a polyedral angle depends only upon the *relative position* of its faces, and is independent of their extent. Thus, by the face SAB is not meant the triangle SAB, but the indefinite plane between the edges SA, SB produced indefinitely.

557. Two polyedral angles are *equal*, when the face and

diedral angles of one are respectively equal to the face and diedral angles of the other, taken in the *same order*.

558. A polyedral angle of three faces is called a *triedral* angle; one of four faces is called a *tetraedral* angle; etc.

559. A polyedral angle is *convex* when its base is a convex polygon. (141)

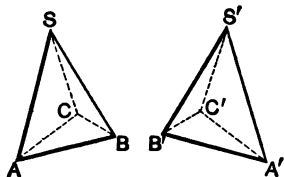
560. A triedral angle is called *isosceles* when it has two of its face-angles equal; when it has all three of its face-angles equal it is called *equilateral*.

561. A triedral angle is called *rectangular*, *bi-rectangular*, or *tri-rectangular*, according as it has one; two, or three, right diedral angles.

The corner of a cube is a tri-rectangular triedral angle.

562. Two polyedral angles are *symmetrical*, when the face and diedral angles of one are equal to the face and diedral angles of the other, each to each, but arranged in *reverse order*.

Thus, the triedral angles $S-ABC$, $S'-A'B'C'$ are symmetrical when the face-angles ASB , BSC , CSA are equal respectively to the face-angles $A'S'B'$, $B'S'C'$, $C'S'A'$, and the diedral angles SA , SB , SC to the diedral angles $S'A'$, $S'B'$, $S'C'$.



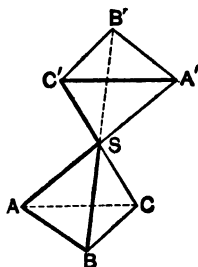
When two polyedral angles are symmetrical, it is, in general, impossible to bring them into coincidence.

The two hands are an illustration. The right hand is symmetrical to the left hand, but cannot be made to coincide with it. The right glove will not fit the left hand, but is symmetrical to it.

563. *Opposite* or *vertical* polyedral angles are those in which the edges of one are the prolongations of the edges of the other.

Proposition 22. Theorem.

564. *Two opposite polyedral angles are symmetrical.*



Hyp. Let $S-ABC, S-A'B'C'$ be two opp. triedral $\angle$ s.

To prove they are symmetrical.

Proof. Because the face $\angle$ s ASB and $A'SB'$ are vertical $\angle$ s,

$$\therefore \angle ASB = \angle A'SB'. \quad (53)$$

Similarly, $\angle BSC = \angle B'SC'$, etc.

Also, because the diedral $\angle$ between two planes is the same at every pt., (528)

$\therefore$ the diedral $\angle$ s whose edges are SA, SB , etc., = respectively the diedral $\angle$ s whose edges are SA', SB' , etc.

But the edges of $S-A'B'C'$ are arranged in the reverse order from the edges of $S-ABC$.

$\therefore S-ABC$ is symmetrical to $S-A'B'C'$. Q.E.D.

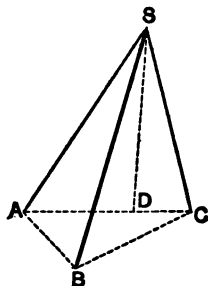
EXERCISE.

Pass two parallel planes, one through each of two straight lines which do not meet and are not parallel.

Let AB, CD be the lines: draw $AE \parallel$ to CD , $CF \parallel$ to AB . $\therefore$ plane AEB is $\parallel$ to plane CFD .

Proposition 23. Theorem.

565. *The sum of any two face-angles of a triedral angle is greater than the third.*



Hyp. Let $S-ABC$ be a triedral $\angle$ in which $\angle ASC$ is the greatest face $\angle$.

To prove $\angle ASB + \angle BSC > \angle ASC$.

Proof. In the plane ASC draw SD , making $\angle ASD = \angle ASB$.

Draw AC cutting SD in D , take $SB = SD$, and join AB , BC .

Because the $\triangle s$ ASB , ASD have

SA common, $SB = SD$, $\angle ASB = \angle ASD$, (Cons.)

$\therefore \triangle ASB = \triangle ASD$, $\therefore AB = AD$. (104)

In $\triangle ABC$, $AB + BC > AC$. (96)

But $AB = AD$. (Proved above)

Subtracting, $BC > DC$. (Ax. 5)

Because in the $\triangle s$ BSC , DSC ,

SC is common, $SB = SD$, and $BC > DC$,

$\therefore \angle BSC > \angle DSC$. (120)

But $\angle ASB = \angle ASD$. (Cons.)

Adding, $\angle ASB + \angle BSC > \angle ASC$. Q.E.D.

Proposition 24. Theorem.

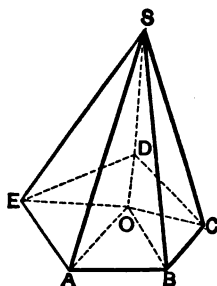
566. *The sum of the face-angles of any convex polyedral angle is less than four right angles.*

Hyp. Let the convex polyedral $\angle S$ be cut by a plane making the section $ABCDE$ a convex polygon.

To prove $\angle ASB + \angle BSC$, etc.,
 $< 4 \text{ rt. } \angle s$.

Proof. From any pt. O within the polygon $ABCDE$ draw OA, OB, OC, OD, OE .

There are thus formed two sets of Δs , one with their common vertex at S , and the other with their common vertex at O , and an equal number of each.



$\therefore$ the sum of the $\angle s$ of these two sets of Δs is equal. (97)

Because the sum of any two face $\angle s$ of a triedral $\angle$
 $>$ the third, (565)

$$\therefore \angle SAE + \angle SAB > \angle EAB,$$

$$\text{and} \quad \angle SBA + \angle SBC > \angle ABC, \text{ etc.}$$

Taking the sum of these inequalities, we find that the sum of the $\angle s$ at the bases of the Δs whose common vertex is $S >$ the sum of the $\angle s$ at the bases of the Δs whose common vertex is O .

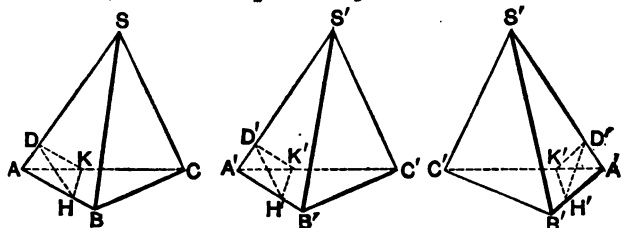
$\therefore$ the sum of the $\angle s$ at $S <$ the sum of the $\angle s$ at O .

$\therefore$ the sum of the $\angle s$ at $S < 4 \text{ rt. } \angle s$. (56)

Q.E.D.

Proposition 25. Theorem.

567. *Two triedral angles, which have the three face-angles of the one equal respectively to the three face-angles of the other, are either equal or symmetrical.*



Hyp. In the triedral $\angle s$, S and S' , let the face $\angle s$ ASB , BSC , CSA = the face $\angle s$ $A'S'B'$, $B'S'C'$, $C'S'A'$.

To prove $S - ABC$ = or symmetrical to $S' - A'B'C'$.

Proof. In the edges SA , $S'A'$, etc., take the six equal distances SA , SB , SC , $S'A'$, $S'B'$, $S'C'$; and join AB , BC , CA , $A'B'$, $B'C'$, $C'A'$.

Then $\triangle s$ SAB , SBC , SCA = $\triangle s$ $S'A'B'$, $S'B'C'$, $S'C'A'$.

$$\therefore \triangle ABC = \triangle A'B'C'. \quad (108)$$

In the edges SA , $S'A'$, take $SD = S'D'$.

In the faces ASB , ASC , and $A'S'B'$, $A'S'C'$, draw DH , DK , and $D'H'$, $D'K'$ $\perp$ to AS and $A'S'$, meeting AB , AC and $A'B'$, $A'C'$, in H , K and H' , K' . Join HK and $H'K'$.

Because $AD = A'D'$, and $\angle DAH = \angle D'A'H'$,

$$\therefore \text{rt. } \triangle ADH = \text{rt. } \triangle A'D'H'. \quad (107)$$

$$\therefore AH = A'H', \text{ and } DH = D'H'.$$

In the same way, $AK = A'K'$, and $DK = D'K'$.

$$\therefore \triangle AHK = \triangle A'H'K', \text{ and } HK = H'K'. \quad (104)$$

$$\therefore \triangle HDK = \triangle H'D'K', \text{ and } \angle HDK = \angle H'D'K'. \quad (108)$$

$$\therefore \text{diedral } \angle SA = \text{diedral } \angle S'A'. \quad (536)$$

Similarly, diedral $\angle s$ SB , SC = diedral $\angle s$ $S'B'$, $S'C'$.

Now if the equal $\angle s$ are arranged in the same order, as in the first two figures, the two triedral $\angle s$ are equal. (557)

But if the equal $\angle s$ are in reverse order, as in the first and third figures, the triedral $\angle s$ are symmetrical. (562)

Q.E.D.

EXERCISES.

THEOREMS.

1. If a straight line is parallel to a plane, any plane perpendicular to the line is perpendicular to the plane.

See (503) and (540).

2. If a line is parallel to each of two intersecting planes, it is parallel to their intersection.

3. If a line is perpendicular to a plane, every plane parallel to that line is also perpendicular to the plane.

4. If a line is parallel to each of two planes, the intersections which any plane passing through it makes with the planes are parallel.

5. If a straight line and a plane are perpendicular to the same straight line, they are parallel.

6. If two straight lines are parallel, they are parallel to the common intersection of any two planes passing through them.

7. If the intersections of several planes are parallel, the normals drawn to them from any point are in one plane.

8. If a plane is passed through one of the diagonals of a parallelogram, the perpendiculars to it from the extremities of the other diagonal are equal.

9. Prove that the sides of an isosceles triangle are equally inclined to any plane through the base.

10. From a point P, PA is drawn perpendicular to a given plane, and from A, AB is drawn perpendicular to a line in that plane: prove that PB is also perpendicular to that line.

11. If the projections of any line upon two intersecting planes are each of them straight lines, prove that the line itself is a straight line.

See (495).

12. If a plane is passed through the middle point of the common perpendicular to two straight lines in space, and parallel to both of these lines, prove that the plane bisects

every straight line which joins a point of one of these lines to a point of the other.

See (526).

13. The projection of a straight line on a plane is a straight line which is either parallel to the first straight line or meets it in the point where it cuts the plane, according as the first line is parallel to the plane or not.

14. From a point P , PA and PB are drawn perpendicular to two planes which intersect in CD , meeting them in A and B ; from A , AE is drawn perpendicular to CD : prove that BE is also perpendicular to CD .

15. Two planes which are not parallel are cut by two parallel planes: prove that the intersections of the first two with the last two contain equal angles.

See (519), (525).

16. If two parallel planes be cut by three other planes which have a point, but no line common to all three, and no two of which are parallel, the triangles formed by the intersections of the parallel planes with the other three planes are similar to each other.

See (519), (525).

17. The projections of parallel straight lines on any plane are themselves parallel.

See (509), (525), (519).

18. Two straight lines which intersect are inclined to each other at an angle equal to two-thirds of a right angle, and to a given plane, each, at an angle equal to half a right angle. Prove that their projections on this plane are at right angles to each other.

19. In any trihedral angle, the planes bisecting the three dihedral angles, all intersect in the same straight line.

See (549).

20. In any trihedral angle, the planes which bisect the three face-angles, and are perpendicular to these faces respectively, all intersect in the same straight line.

From the vertex S , take equal distances SA , SB , SC , on the three edges; the intersections of the three $\perp$ bisecting planes with the plane of ABC are $\perp$ bisecting lines of the sides of ABC , and have a common intersection (168); $\therefore$ etc.

21. In any triedral angle, the three planes which pass through the edges, perpendicular to the opposite faces respectively, all intersect in the same straight line.

See (546).

22. ABCD is a face and AE a diagonal of a cube, BG is drawn perpendicular to AE, and DG is joined: prove that DG is perpendicular to AE.

23. If two face-angles of a triedral angle are equal, the dihedral angles opposite them are also equal.

24. An isosceles triedral angle and its symmetrical triedral angle are equal.

25. If BAC, CAD, DAB be the three face-angles of a triedral angle, prove that the angle between AD and the straight line bisecting the angle BAC is less than half the sum of the angles BAD, CAD.

See (565).

26. A triedral angle is contained by the three face-angles BOC, COA, AOB; if BOC, COA are together equal to two right angles, prove that CO is perpendicular to the line which bisects the angle AOB.

LOCI.

27. Find the locus of points which are equally distant from three given points not in the same straight line.

28. Find the locus of points which are equally distant from two given intersecting straight lines.

29. Find the locus of points which are equally distant from two given parallel planes; or whose distances from the parallel planes are in a given ratio.

30. Find the locus of points which are equally distant from three given planes.

See (549).

31. Find the locus of points which are equally distant from the three edges of a triedral angle.

32. Find the locus of points which are equally distant from the three faces of a triedral angle.

33. Find the locus of points which are equally distant

from two given planes, and at the same time equally distant from two given points.

34. Find the locus of a point such that the sum of its distances from two given planes is equal to a given straight line.

35. Find the locus of a point such that the sum of its distances from three given planes is equal to a given straight line.

PROBLEMS.

36. Pass a plane perpendicular to a given straight line through a given point not in that line.

37. Pass a plane through a given straight line, perpendicular to a given plane.

See (543).

38. Pass a plane through a given point parallel to a given plane.

See (522).

39. Find the point in a given straight line which is equally distant from two given points not in the same plane with the given line.

40. Draw a straight line through a given point in space, so that it shall cut two given straight lines not in the same plane.

41. Draw a straight line through a given point in a given plane, so that it shall be perpendicular to a given line in space.

42. Two given straight lines do not intersect and are not parallel: find a plane on which their projections will be parallel.

43. Divide a straight line similarly to a given divided straight line lying in a different plane.

Let ACDB be the divided line, NMLF the other line; draw EHKG $\parallel$ to AB, and CH, DK, BG $\parallel$ to AE; also KL, HM, EN $\parallel$ to GF: $\therefore$ (525), (526).

44. Given three straight lines meeting at a point: draw through the given point a straight line equally inclined to the three.

BOOK VII

POLYEDRONS.

DEFINITIONS.

568. A *polyedron* is a solid bounded by planes. The portions of the bounding planes, limited by their mutual intersections, are the *faces* of the polyedron; the intersections of the faces are the *edges*, and the intersections of the edges are the *vertices*, of the polyedron.

A *diagonal* of a polyedron is a straight line joining two vertices not in the same face.

569. A polyedron of four faces is called a *tetraedron*; one of six faces, a *hexaedron*; one of eight faces, an *octaedron*; one of twelve, a *dodecaedron*; one of twenty, an *icosaedron*.

570. A polyedron is *convex* when the section made by any plane intersecting it is a convex polygon.

Only convex polyedrons are treated of in this work.

571. The *volume* of a solid is its numerical measure (229), referred to some other solid called the *unit of volume*.

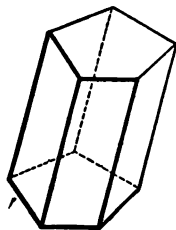
572. Two solids are *equivalent* when they have equal volumes.

573. Two polygons are said to be *parallel* when their sides are respectively parallel.

PRISMS AND PARALLELOPIPEDS.

574. A *prism* is a polyedron two of whose faces are equal and parallel polygons, and the other faces are parallelograms.

The equal and parallel polygons are called the *bases* of the prism; the parallelograms are the *lateral faces*; the lateral faces taken together form the *lateral* or *convex surface*; and the intersections of the lateral faces are the *lateral edges*.



The lateral edges are parallel and equal (511) and (520). The area of the lateral surface is called the *lateral area*.

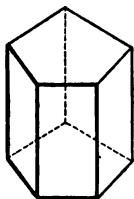
575. The *altitude* of a prism is the perpendicular distance between its bases.

576. A prism is said to be *triangular*, *quadrangular*, *hexagonal*, etc., according as its bases are triangles, quadrilaterals, hexagons, etc.

577. A *right prism* is one whose lateral edges are perpendicular to its bases.

A *regular prism* is a right prism whose bases are regular polygons: therefore its lateral faces are equal rectangles.

An *oblique prism* is one whose lateral edges are not perpendicular to its bases.

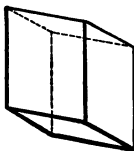


578. A *right section* of a prism is a section by a plane perpendicular to its lateral edges.



579. A *truncated prism* is the part of a prism included between the base and an oblique section made by a plane cutting all the lateral edges.

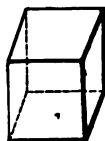
580. A *parallelepiped* is a prism whose bases are parallelograms: hence all the faces are parallelograms.



581. A *right parallelepiped* is one whose lateral edges are perpendicular to the bases: hence the lateral faces are rectangles.

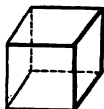
582. A *rectangular parallelopiped* is one whose faces are all rectangles.

The three edges of a parallelopiped which meet at a vertex are its *dimensions*.



583. A *cube* is a rectangular parallelopiped whose six faces are all squares.

584. The cube whose edge is the unit of length is taken as the unit of volume. (571)



REM. In a *right* parallelopiped the adjacent lateral faces may make any angle with each other, while in a *rectangular* parallelopiped the faces are perpendicular to each other.

Proposition 1. Theorem.

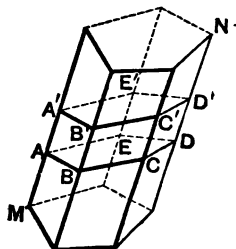
585. The sections of the lateral faces of a prism made by parallel planes are equal polygons.

Hyp. Let the prism MN be cut by the $\parallel$ planes AD, A'D'.

To prove $ABCDE = A'B'C'D'E'$.

Proof. Because the intersections of two $\parallel$ planes by a third plane are parallel, (519)

$\therefore$ AB, BC, CD, etc., are $\parallel$ respectively to A'B', B'C', C'D', etc.



$$\therefore \angle ABC = \angle A'B'C', \angle BCD = \angle B'C'D'. \quad (525)$$

Because $\parallel$ s included between $\parallel$ s are equal, (131)

$$\therefore AB = A'B', BC = B'C', \text{ etc.}$$

$\therefore$ the polygons ABCDE, A'B'C'D'E' are mutually equilateral and mutually equiangular.

$$\therefore ABCDE = A'B'C'D'E'. \quad (146)$$

Q. E. D.

586. COR. Any section of a prism by a plane parallel to the base is equal to the base; and all right sections are equal.

Proposition 2. Theorem.

587. *If three faces including a triedral angle of a prism are equal respectively to three faces including a triedral angle of a second prism, and similarly placed, the two prisms are equal.*

Hyp. Let the faces AD , AG , BH , of the triedral $\angle B$, be = respectively to the faces $A'D'$, $A'G'$, $B'H'$ of the triedral $\angle B'$.

To prove

prism AI = prism $A'I'$.

Proof. Since the face $\angle s ABC, ABG, CBG$ = the face $\angle s A'B'C', A'B'G', C'B'G'$, respectively, (Hyp.)

$\therefore$ triedral $\angle B$ = triedral $\angle B'$. (567)

$\therefore$ the prism AI may be applied to the prism $A'I'$ so that the base AD shall coincide with $A'D'$, the face AG with $A'G'$, and the face BH with $B'H'$, the pts. D, E falling on D', E' .

Since the pts. F, G, H coincide with F', G', H' ,

$\therefore$ the planes of the upper bases will coincide. (484)

Since the lateral edges of the prisms are equal and parallel, (574)

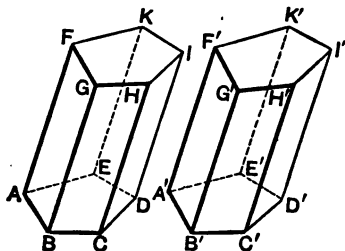
$\therefore$ the edges DI, EK will coincide with $D'I', E'K'$, respectively, and the pts. I, K with the pts. I', K' .

$\therefore$ the prisms coincide throughout, and $\therefore$ are equal.

Q.E.D.

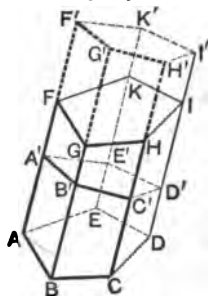
588. COR. 1. *Two right prisms which have equal bases and equal altitudes are equal.*

589. COR. 2. *Two truncated prisms are equal if three faces including a triedral angle of the one are equal respectively to three faces including a triedral angle of the other, and similarly placed.*



Proposition 3. Theorem.

590. *An oblique prism is equivalent to a right prism having for its base a right section of the oblique prism and for its altitude a lateral edge of the oblique prism.*



Hyp. Let $ABCDE-I$ be an oblique prism, and $A'D'$ a rt. section of it.

Produce AF to F' , making $A'F' = AF$, and through F' pass the plane $F'I' \perp$ to AF' , cutting all the edges BG , CH , etc., produced, in the pts. G' , H' , etc., and forming the rt. section $F'I' \parallel$ to $A'D'$. (516)

To prove prism $AI =$ prism $A'I'$.

Proof. In the truncated prisms AD' and FI' ,

$$ABCDE = FGHIK. \quad (516)$$

Because $AF = A'F'$, and $BG = B'G'$,

$$\therefore AA' = FF', \text{ and } BB' = GG'.$$

Because AG and $A'G'$ are $\square$ s, (574)

$\therefore AB$ is $=$ and $\parallel$ to FG , and $A'B'$ is $=$ and $\parallel$ to $F'G'$.

$$\therefore \text{quadr. } AB' = \text{quadr. } FG',$$

being mutually equilateral and equiangular (146).

Similarly, $BC' = GH'$.

$\therefore$ the truncated prisms AD' and FI' are equal. (589)

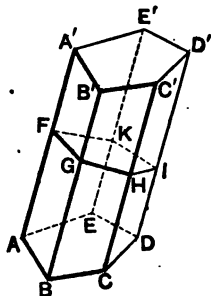
Taking AD' from the whole solid, the right prism $A'I'$ remains, and taking FI' from the same solid, the oblique prism AI remains.

$$\therefore \text{prism } AI = \text{prism } A'I'.$$

Q.E.D.

Proposition 4. Theorem.

591. *The lateral area of a prism is equal to the product of the perimeter of a right section by a lateral edge.*



Hyp. Let $FGHIK$ be a rt. section, and AA' a lateral edge of the prism AD' .

To prove lateral area of

$$AD' = AA' (FG + GH + \text{etc.}).$$

Proof. Since the rt. section FI is $\perp$ to the lateral edges, (578)

$\therefore FG, GH, \text{etc.},$ are $\perp$ to $AA', BB', \text{etc.},$ (487)

and $\therefore FG, GH, \text{etc.},$ are the altitudes of the $\square$ s $AB', BC', \text{etc.},$ which form the lateral area of the prism.

$$\therefore \text{area } AA'B'B = AA' \times FG, \quad (363)$$

and $\therefore \text{area } BB'C'C = BB' \times GH, \text{ etc.}$

Because $AA' = BB' = \text{etc.},$ (574), and $FG + GH + \text{etc.}, =$ the perimeter of the rt. section, $\therefore$ adding the above areas, we have

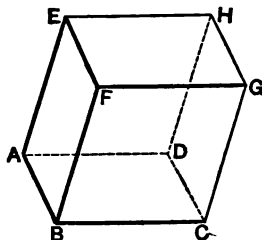
$$\begin{aligned} \text{lateral area } AD' &= AA' \times FG + AA' \times GH + \text{etc.} \\ &= AA'(FG + GH + HI + \text{etc.}). \end{aligned}$$

Q. E. D.

592. COR. *The lateral area of a right prism is equal to the product of the perimeter of its base by its altitude.*

Proposition 5. Theorem.

593. *The opposite faces of a parallelopiped are equal and parallel.*



Hyp. Let ABCD-G be a parallelopiped.

To prove face AF = and $\parallel$ to DG.

Proof. Since AC is a $\square$, (580)

$\therefore AB =$ and $\parallel$ to DC.

Since BG is a $\square$, (580)

$\therefore BF =$ and $\parallel$ to CG.

Since BA and BF are $\parallel$ respectively to CD and CG,

$\therefore \angle ABF = \angle DCG$, and plane AF is $\parallel$ to plane DG. (525)

$\therefore \square AF = \square DG$,

having two sides and the included $\angle$ equal, each to each (135).

$\therefore$ face AF = and $\parallel$ to face DG.

It may be proved in the same way that

face AH = and $\parallel$ to face BG.

Also, face AC = and $\parallel$ to face EG, by definition. (574)

Q.E.D.

594. SCH. Since the opposite faces of a parallelopiped are equal and parallel parallelograms, any face may be taken for the base.

595. COR. The twelve edges of a parallelopiped may be divided into three sets, each set having four equal and parallel lines.

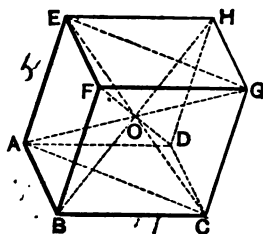
Proposition 6. Theorem.

596. *The four diagonals of a parallelopiped bisect one another.*

Hyp. Let ABCD-G be a parallelopiped.

To prove that the four diagonals AG, BH, CE, DF bisect one another.

Proof. Through the opp. and $\parallel$ edges AE, CG pass a plane: Join AC, EG.



Because AE and CG are $=$ and $\parallel$,

(574)

$\therefore$ the figure ACGE is a $\square$.

$\therefore$ its diagonals AG, EC bisect each other in the pt. O. (134)

In the same way it may be shown that AG, BH, and AG, DF, bisect each other at O.

$\therefore$ the four diagonals bisect one another at the pt. O.

Q.E.D.

597. COR. 1. *The diagonals of a rectangular parallelopiped are equal.*

598. COR. 2. *The square of a diagonal of a rectangular parallelopiped is equal to the sum of the squares of the three edges meeting at any vertex.*

For, if AG is a rectangular parallelopiped, the rt. $\triangle$ s ACG, ABC give

$$\overline{AG}^2 = \overline{AC}^2 + \overline{CG}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{BF}^2.$$

599. SCH. The point O through which all four diagonals pass is called the *centre* of the parallelopiped.

Proposition 7. Theorem.

600. *The plane passed through two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms.*

Hyp. Let the parallelopiped ABCD-G be divided by the plane AEGC, passing through the opp. edges AE and CG.

To prove that the triangular prisms ABC-G and ADC-G are equivalent.

Proof. Take a rt. section KLMO of the parallelopiped, intersecting the plane AEGC in the line KM.

Because the faces AF and AH are $\parallel$ respectively to DG and BG,

$$\therefore KL \text{ is } \parallel \text{ to } OM, \text{ and } KO \text{ is } \parallel \text{ to } LM. \quad (519)$$

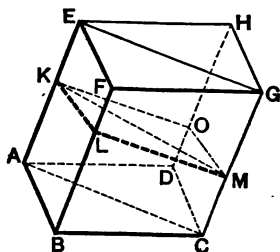
$$\therefore KLMO \text{ is a } \square. \quad (124)$$

$\therefore$ the intersection KM of the plane AEGC with the rt. section is the diagonal of the $\square$ KLMO, and divides it into two equal $\triangle$ s KLM, KOM. (130)

The oblique prism ABC-E is equivalent to a rt. prism whose base is KLM and altitude AE, and the oblique prism ADC-E is equivalent to a rt. prism whose base is KOM and alt. AE. (590)

But since these two rt. prisms have equal bases KLM and KOM, and the same altitude, they are equal. (588)

$\therefore$ prism ABC-G is equivalent to prism ADC-G. Q.E.D.



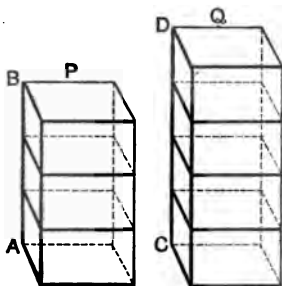
Proposition 8. Theorem.

601. *Two rectangular parallelopipeds having equal bases are to each other as their altitudes.*

Hyp. Let AB and CD be the altitudes of two rectangular parallelopipeds, P and Q, having equal bases.

To prove $P : Q = AB : CD$.

Proof. Suppose the altitudes AB and CD have a common measure, which is contained m times in AB and n times in CD.



Then $AB : CD = m : n$.

Divide AB and CD by this common measure.

Through the several pts. of division of AB and CD pass planes $\perp$ to these lines.

The parallelopiped P will be divided into m , and Q into n , parallelopipeds, all equal to one another. (588)

$$\therefore P : Q = m : n;$$

and $\therefore P : Q = AB : CD$. Q.E.D.

The case in which the altitudes AB and CD are incommensurable, is included in this demonstration by the reasoning in (232); or the proof may be extended as in (234), (298), and (356).

602. SCH. This theorem may also be expressed as follows:

Two rectangular parallelopipeds which have two dimensions in common are to each other as their third dimensions.

Proposition 9. Theorem.

603. *Two rectangular parallepipeds having equal altitudes are to each other as their bases.*

Hyp. Let the rectangles ab and $a'b'$ be the bases of two rectangular parallepipeds, P and Q , having the common altitude c .

To prove
$$\frac{P}{Q} = \frac{ab}{a'b'}.$$

Proof. Construct a third parallelepiped R whose dimensions are a , b' , and c .

Then, because P and R have the two dimensions a and c in common,

(Cons.)

$$\therefore \frac{P}{R} = \frac{b}{b'}. \quad (602)$$

And because R and Q have the two dimensions b' and c in common,

(Cons.)

$$\therefore \frac{R}{Q} = \frac{a}{a'}. \quad (602)$$

Multiplying these equalities, we have

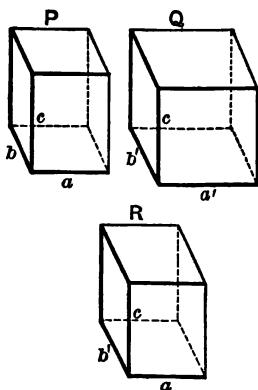
$$\frac{P}{Q} = \frac{ab}{a'b'}. \quad \text{Q.E.D.}$$

604. SCH. This theorem may also be expressed as follows:

Two rectangular parallepipeds having one dimension in common are to each other as the products of the other two dimensions.

EXERCISE.

Find the length of the diagonal of a rectangular parallelepiped whose edges are 1, 4, and 8.



Proposition 10. Theorem.

605. Any two rectangular parallelopipeds are to each other as the products of their three dimensions.

Hyp. Let P and Q be two rectangular parallelopipeds whose dimensions are a, b, c , and a', b', c' , respectively.

To prove
$$\frac{P}{Q} = \frac{abc}{a'b'c'}.$$

Proof. Make a third rectangular parallelopiped R whose dimensions are a', b' , and c .

Then, because P and R have the dimension c in common,

$$\therefore \frac{P}{R} = \frac{ab}{a'b'}. \quad (604)$$

And because R and Q have the two dimensions a', b' in common,

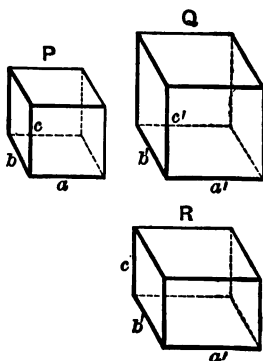
$$\therefore \frac{R}{Q} = \frac{c}{c'}. \quad (602)$$

Multiplying these equalities, we have

$$\frac{P}{Q} = \frac{abc}{a'b'c'}. \quad \text{Q.E.D.}$$

EXERCISE.

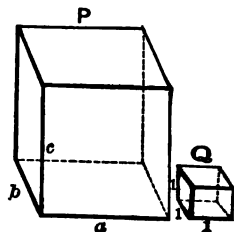
Find the ratio of two rectangular parallelopipeds whose dimensions are 4, 7, 9, and 6, 14, 15, respectively.



Proposition 11. Theorem.

606. *The volume of a rectangular parallelepiped is equal to the product of its three dimensions.*

Hyp. Let P be the rectangular parallelepiped, a , b , and c its dimensions, and let Q be the cube whose edge is the linear unit. Q then is the unit of volume. (584)



To prove $P = abc$.

$$\text{Proof.} \quad \frac{P}{Q} = \frac{a \times b \times c}{1 \times 1 \times 1} = abc. \quad (605)$$

But, since Q is the unit of volume,

$$\therefore \frac{P}{Q} = \text{the volume of } P. \quad (571)$$

$\therefore$ volume of $P = abc$.

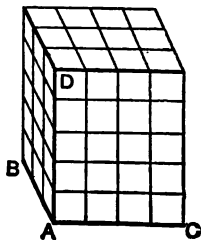
Q. E. D.

607. SCH. The statement of this theorem is an abbreviation of the following:

The number of units of volume in a rectangular parallelepiped is equal to the product of the numbers which measure the linear units in its three dimensions.

Compare (361).

When the three dimensions of a rectangular parallelepiped are each exactly divisible by the linear unit, the truth of the theorem may be shown by dividing the solid into cubes, each equal to the unit of volume. Thus, if AB contain the linear unit 3 times, AC , 4 times, and AD , 5 times, these edges may be divided respectively into 3, 4, and 5 equal parts, and then planes passed through the several points of division at right angles to these edges will divide the solid into cubes each equal to the unit



of volume. Hence the whole solid contains $3 \times 4 \times 5$, or 60 cubes, each equal to the unit of volume.

608. COR. 1. Since $a \times b$ is the area of the base, and c is the altitude, of the parallelopiped P; therefore the above result may be expressed in the form:

The volume of a rectangular parallelopiped is equal to the product of its base and altitude.

609. COR. 2. *The volume of a cube is the third power of its edge*, being the product of three equal factors; if the edge is 1, the volume is $1 \times 1 \times 1 = 1$; if the edge is a , the volume is $a \times a \times a = a^3$. Hence it is that in Arithmetic and Algebra the “cube” of a number is the name given to the “third power” of a number.

EXERCISES.

1. Find the surface, and also the volume, of a rectangular parallelopiped whose edges are 4, 7, and 9 feet.

2. Find the surface of a rectangular parallelopiped whose base is 8 by 12 feet and whose volume is 384 cubic feet.

3. Find the volume of a rectangular parallelopiped whose surface is 208 and whose base is 4 by 6.

4. Find the length of the diagonal of a rectangular parallelopiped whose edges are 3, 4, and 5.

5. Find the ratio of two rectangular parallelopipeds whose dimensions are 1, 4, 8, and 3, 4, 5, respectively.

6. Find the surface of a rectangular parallelopiped whose base is 7 by 9 feet and whose volume is 315 cubic feet.

7. Find the volume of a rectangular parallelopiped whose surface is 416 and whose base is 4 by 12.

8. A man wishes to make a cubical cistern whose contents are 186624 cubic inches: how many feet of inch boards will line it?

9. Find the side of a cube which contains as much as a rectangular parallelopiped 20 feet long, 10 feet wide, and 6 feet high.

Proposition 12. Theorem.

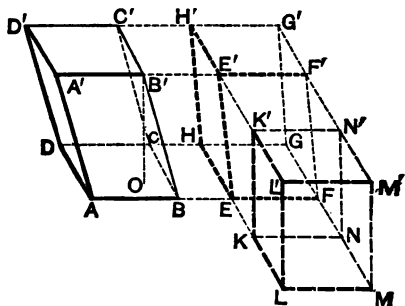
610. *The volume of any parallelopiped is equal to the product of its base and altitude.*

Hyp. Let $B'O$ be the altitude of the oblique parallelopiped AC' .

To prove that
vol. $AC' = ABCD \times B'O$.

Proof. Produce the edges AB , DC , $A'B'$, $D'C'$, and take

$$EF = AB.$$



The rt. parallelopiped EG' , formed by the sections $EE'H'H$, $FF'G'G \perp$ to the produced edge EF , is equivalent to AC' . (590)

Now produce the edges HE , GF , $G'F'$, $H'E'$, and take $KL = HE$.

The parallelopiped $KNN'K'-L$, formed by the sections $LMM'L'$, $KNN'K' \perp$ to the produced edge KL , is equivalent to EG' , and $\therefore$ to AC' . (590)

$\therefore$ given parallelopiped AC' and the last, KM' , are equivalent.

But, since the rt. sections LM' , KN' , are rectangles,

$$\therefore KM' \text{ is a rectangular parallelopiped. } (582)$$

Also, since

area $ABCD = \text{area } EFGH = \text{area } KLMN$, (363) (360),
and the three solids have the same altitude $B'O$, (523)

$$\therefore \text{ the volume } KM' = KLMN \times B'O. (608)$$

$$\therefore \text{ the volume } AC' = ABCD \times B'O. \text{ Q.E.D.}$$

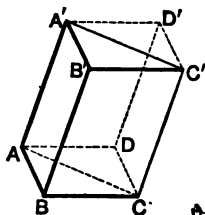
Proposition 13. Theorem.

611. *The volume of a triangular prism is equal to the product of its base and altitude.*

Hyp. Let H denote the altitude of the triangular prism $ABC-B'$.

To prove $\text{vol. } ABC-B' = ABC \times H$.

Proof. Complete the parallelopiped $ABCD-D'$, having its edges = and $\parallel$ to AB, BC, BB' .



$$\text{Then} \quad \text{vol. } ABC-B' = \frac{1}{3} \text{ vol. } ABCD-B', \quad (600)$$

$$\text{and} \quad \text{area } ABC = \frac{1}{3} \text{ area } ABCD. \quad (130)$$

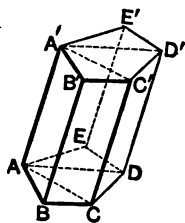
$$\text{But} \quad \text{vol. } ABCD-D' = ABCD \times H. \quad (610)$$

$$\therefore \text{vol. } ABC-B' = \frac{1}{3} ABCD \times H \\ = ABC \times H.$$

Q. E. D.

612. COR. 1. *The volume of any prism is equal to the product of its base and altitude.*

For, any prism may be divided into triangular prisms by passing planes through a lateral edge AA' and the corresponding diagonals of the base. Then the volume of the given prism is the sum of the volumes of the triangular prisms, that is, the sum of their bases, or the base of the given prism, multiplied by the common altitude.



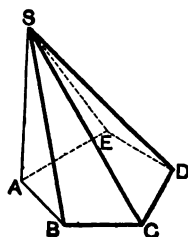
613. COR. 2. *Two prisms are to each other as the products of their bases and altitudes; two prisms having equivalent bases are to each other as their altitudes; two prisms of the same altitude are to each other as their bases; two prisms having equivalent bases and equal altitudes are equivalent.*

PYRAMIDS.

DEFINITIONS.

614. A *pyramid* is a polyedron bounded by a polygon, and by triangles meeting at a common point, the sides of the polygon being the bases of the triangles.

The polygon $ABCDE$ is called the *base* of the pyramid; the point S , in which the triangles meet, is called the *vertex*; the triangular faces are called the *lateral faces*, and taken together they form the *lateral*, or *convex surface*; the intersections SA , SB , etc., of the lateral faces are called the *lateral edges*; and the area of the lateral surface is called the *lateral area*.



615. The *altitude* of a pyramid is the perpendicular distance from the vertex to the plane of the base.

616. A pyramid is called *triangular*, *quadrangular*, *pentagonal*, etc., according as its base is a triangle, quadrilateral, pentagon, etc.

617. A triangular pyramid has but four faces, and is called a *tetradron*; any one of its faces can be taken for its base.

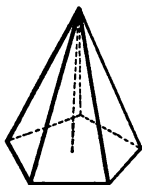
NOTE.—The six edges of a triangular pyramid may be divided into three pairs, such that the two edges of a pair do not meet each other. Since each edge meets two other edges at one vertex, and yet another two edges at the adjoining vertex, there is but one edge left to pair with it. The pair is called a pair of opposite edges.

618. A *regular pyramid* is one whose base is a regular polygon, the centre of which coincides with the foot of the

perpendicular let fall on it from the vertex. This perpendicular is called the *axis* of the pyramid.

619. The lateral edges of a regular pyramid are equal (496); therefore the lateral faces are equal isosceles triangles.

620. The *slant height* of a regular pyramid is the perpendicular distance from the vertex to the base of any one of its lateral faces.



621. A *frustum of a pyramid* is the portion of a pyramid included between its base and a plane parallel to the base, and cutting all the lateral edges. If the cutting plane is not parallel to the base, the portion included between the base and plane is called a *truncated pyramid*.

622. The *altitude* of a frustum is the perpendicular distance between its bases.

In a frustum of a regular pyramid the lateral faces are equal trapezoids.

The *slant height* of the frustum of a regular pyramid is the perpendicular distance between the parallel sides of one of these trapezoids.

EXERCISES.

1. Show that the slant height of a regular pyramid is the straight line drawn from the vertex of the pyramid to the middle point of any side of the base.
2. Show that the lateral faces of a frustum of a regular pyramid are equal trapezoids.
3. Find the volume of a right triangular prism if the height is 24 inches and the sides of the base are 24, 20, and 20 inches.
4. Find the volume of a triangular prism if the height is 18 inches and the sides of the base are 6, 8, and 10 inches.

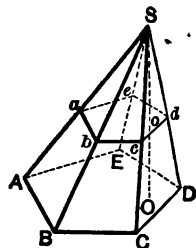
Proposition 14. Theorem.

623. *If a pyramid is cut by a plane parallel to its base:*

(1) *The edges and the altitude are divided proportionally.*

(2) *The section is a polygon similar to the base.*

Hyp. Let S-ABCDE be cut by the plane $abcde \parallel$ to the base, intersecting the lateral edges in a, b, c, d, e , and the altitude in o .



(1) *To prove* $\frac{Sa}{SA} = \frac{Sb}{SB} \dots = \frac{So}{SO}$

Proof. Suppose a plane passed through S $\parallel$ to the base.

Then, because the edges and altitude are cut by three $\parallel$ planes,

$$\therefore \frac{Sa}{SA} = \frac{Sb}{SB} = \dots = \frac{So}{SO} \quad (526)$$

(2) *To prove section abcde similar to ABCDE.*

Proof. Because $ab \parallel$ to AB , $bc \parallel$ to BC , $cd \parallel$ to CD , etc., (519)

$$\therefore \angle abc = \angle ABC, \angle bcd = \angle BCD, \text{ etc.} \quad (525)$$

Also, since the $\triangle Sab$, Sbc , etc., are similar respectively to the $\triangle SAB$, SBC , etc., being mutually equiangular,

$$\therefore \frac{ab}{AB} = \frac{Sb}{SB}, \text{ and } \frac{bc}{BC} = \frac{Sb}{SB}; \therefore \frac{ab}{AB} = \frac{bc}{BC}$$

In like manner, $\frac{bc}{BC} = \frac{cd}{CD}$, etc.

$\therefore$ the polygons $abcde$ and $ABCDE$ are mutually equiangular and have their homologous sides proportional.

$\therefore$ the section $abcde$ is similar to the base $ABCDE$.

Q.E.D.

624. COR. 1. Because $abcde$ and $ABCDE$ are similar polygons,

$$\therefore \frac{abcde}{ABCDE} = \frac{\overline{ab}^2}{\overline{AB}^2} = \frac{\overline{Sb}^2}{\overline{SB}^2} = \frac{\overline{So}^2}{\overline{SO}^2}. \quad (379)$$

Hence: *The areas of parallel sections of a pyramid are proportional to the squares of their distances from the vertex.*

625. COR. 2. *If two pyramids of equal altitudes are cut by planes parallel to their bases, and at equal distances from their vertices, the sections will be proportional to the bases.*

For,
$$\frac{abcde}{ABCDE} = \frac{\overline{So}^2}{\overline{SO}^2}. \quad (624)$$

Similarly,
$$\frac{a'b'c'}{A'B'C'} = \frac{\overline{S'o'}^2}{\overline{S'O'}^2},$$

where $A', B', C', a', b', c', S', O', o'$, are the corresponding pts. of the second pyramid.

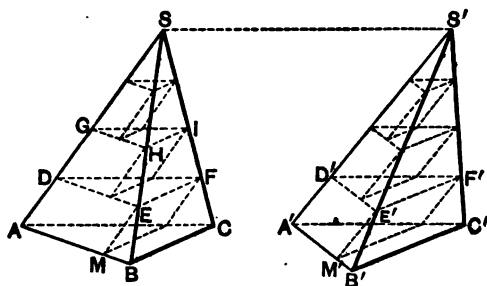
But $So = S'o'$, and $SO = S'O'$.

$$\therefore abcde : ABCDE = a'b'c' : A'B'C'.$$

626. COR. 3. *In two pyramids of equal altitudes and equivalent bases, sections made by planes parallel to their bases and at equal distances from their vertices are equivalent.*

Proposition 15. Theorem.

627. *Two triangular pyramids having equivalent bases and equal altitudes are equivalent.*



Hyp. Let $S-ABC$ and $S'-A'B'C'$ have equivalent bases ABC , $A'B'C'$ in the same plane and a common altitude.

To prove $\text{vol. } S-ABC = \text{vol. } S'-A'B'C'$.

Proof. Divide the common altitude into any number of equal parts, and through the pts. of division pass planes $\parallel$ to the plane of the bases.

In the pyramid $S-ABC$, on the sections DEF , GHI , etc., as upper bases, construct prisms with lateral edges $=$ and $\parallel$ to AD .

Similarly, construct prisms on the sections of $S'-A'B'C'$, as upper bases.

Since the corresponding sections are equivalent, (626)

$\therefore$ each prism in $S-ABC$ is equivalent to its corresponding prism in $S'-A'B'C'$. (613)

$\therefore$ the sum of the prisms inscribed in the pyramid S-ABC is equivalent to the sum of the prisms inscribed in the pyramid S'-A'B'C'.

Now let the number of equal parts into which the common altitude is divided be increased indefinitely; the sum of the prisms DFE-M, etc., will approach the volume of the pyramid S-ABC as its limit, and the sum of the prisms D'F'E'-M', etc., will approach the volume of the pyramid S'-A'B'C' as its limit.

$\therefore$ ultimately, the volumes of the prisms will differ from the volumes of the pyramids by less than any assignable quantity.

$$\therefore \text{vol. S-ABC} = \text{vol. S'-A'B'C'}.$$

Q.E.D.

Proposition 16. Theorem.

628. *The lateral area of a regular pyramid is equal to half the product of the perimeter of its base by its slant height.*

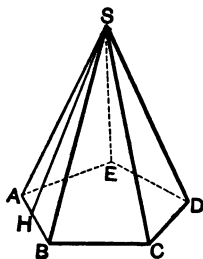
Hyp. Let SH be the slant height of the regular pyramid S-ABCDE.

To prove

$$\text{area S-ABCDE} = \frac{1}{2}(AB + BC + \text{etc.})SH.$$

Proof. Since the lateral area = the sum of the equal isosceles $\triangle$ s SAB, SBC, etc.,

(619)



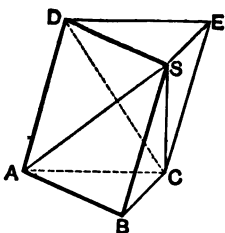
$$\therefore \text{area S-ABCDE} = \frac{1}{2}(AB + BC + \text{etc.})SH. \quad (366)$$

Q.E.D.

629. COR. *The lateral area of the frustum of a regular pyramid, is equal to half the sum of the perimeters of its bases multiplied by the slant height of the frustum.*

Proposition 17. Theorem.

630. *A triangular pyramid is one-third of a triangular prism of the same base and altitude.*



Hyp. Let S-ABC be a triangular pyramid, and ABC-DSE a triangular prism having the same base and altitude.

To prove

$$\text{vol. S-ABC} = \frac{1}{3} \text{ vol. ABC-DSE.}$$

Proof. The prism ABC-D is composed of the triangular pyramid S-ABC and the quadrangular pyramid S-ACED.

Pass the plane SCD through SC and SD, dividing the quadrangular pyramid into the two triangular pyramids S-ACD and S-ECD.

Since the Δ s ACD and ECD are equal, *a triangle of a*
prism divides (130)

$\therefore$ the pyramids S-ACD and S-ECD have equal bases.

Also, they have the same altitude. (615)

$$\therefore \text{vol. S-ACD} = \text{vol. S-ECD.} \quad (627)$$

The pyramid S-ECD may be regarded as having its vertex at C and DSE for its base.

Because, then, the pyramids S-ABC and C-DSE have equal bases (574), and the same altitude, (523)

$$\therefore \text{vol. S-ABC} = \text{vol. C-DSE.} \quad (627)$$

$\therefore$ the three pyramids into which the prism is divided are equivalent to each other.

$$\therefore \text{vol. S-ABC} = \frac{1}{3} \text{ vol. ABC-DSE.} \quad \text{Q.E.D.}$$

631. COR. *The volume of a triangular pyramid is equal to one-third the product of its base and altitude.*

Proposition 18. Theorem.

632. *The volume of any pyramid is equal to one-third the product of its base and altitude.*

Hyp. Let S-ABCDE be any pyramid; denote its volume by V and its altitude by H.

To prove $V = \frac{1}{3}ABCDE \times H$.

Proof. Pass the planes SAC, SAD, etc., through the edge SA and the diagonals AC, AD, etc., dividing the pyramid into triangular pyramids.

The bases of these pyramids are the Δ s which compose the base of the given pyramid, and their common altitude is the altitude H of the pyramid.

Then, $\text{vol. S-ABC} = \frac{1}{3}ABC \times H$,

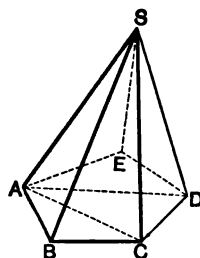
and $\text{vol. S-ACD} = \frac{1}{3}ACD \times H$, etc. (631)

But the volume of the given pyramid = the sum of the volumes of the triangular pyramids.

$$\therefore V = \frac{1}{3}ABCDE \times H. \quad \text{Q.E.D.}$$

633. COR. *Two pyramids are to each other as the products of their bases and altitudes; two pyramids having equivalent bases are to each other as their altitudes; two pyramids having the same altitude are to each other as their bases; two pyramids having equivalent bases and equal altitudes are equivalent.*

634. SCH. The volume of any polyedron may be found by dividing it into pyramids, and computing the volume of each pyramid.



Proposition 19. Theorem.

635. *A frustum of a triangular pyramid is equivalent to the sum of three pyramids, having the same altitude as the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the bases, of the frustum.*

Hyp. Let B denote the lower base, b the upper base, h the altitude, and V the volume of the frustum $ABC-D$.

To prove $V = \frac{h}{3}(B + b + \sqrt{Bb}).$

Proof. Pass the planes AEC , DEC through the pts. A , E , C , and D , E , C , dividing the frustum into three triangular pyramids.

Denote the volume of $E-ABC$, $C-DEF$, $E-ACD$ by P , Q , R , respectively.

The pyramid P has for its base the lower base B of the frustum, and for its altitude the altitude h of the frustum.

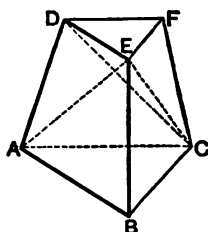
$$\therefore (1) P = \frac{1}{3}B \times h. \quad (632)$$

The pyramid Q has for its base the upper base b of the frustum, and for its altitude the altitude of the frustum.

$$\therefore (2) Q = \frac{1}{3}b \times h. \quad (632)$$

Since the pyramids P and R , regarded as having the common vertex C , and their bases, AEB and AED , in the same plane BD , have a common altitude,

$$\therefore P : R = \triangle AEB : \triangle AED. \quad (633)$$



And since the $\triangle$ s AEB, AED have for their common altitude the altitude of the trapezoid ABED, they are to each other as their bases. (369)•

$$\therefore \frac{P}{R} = \frac{\triangle AEB}{\triangle AED} = \frac{AB}{DE}.$$

In like manner since the pyramids R and Q, regarded as having the common vertex E, and their bases, ACD and CDF, in the same plane AF, have a common altitude,

$$\therefore R : Q = \triangle ACD : \triangle CDF. \quad (633)$$

And since the $\triangle$ s ACD, CDF have for their common altitude the altitude of the trapezoid ACFD, they are to each other as their bases. (369)

$$\therefore \frac{R}{Q} = \frac{\triangle ACD}{\triangle CDF} = \frac{AC}{DF}.$$

Since $\triangle$ s ABC and DEF are similar, (623)

$$\therefore AB : DE = AC : DF. \quad (307)$$

$$\therefore P : R = R : Q.$$

$$\therefore R^2 = P \times Q = \left(\frac{1}{3}\right)h^2 \times B \times b. \text{ From (1) and (2)}$$

$$\therefore R = \frac{1}{3}h \sqrt{B \times b}.$$

$\therefore$ the third pyramid R is equivalent to a pyramid having the same altitude h as the frustum, and for its base the mean proportional between the two bases B and b .

Since the given frustum is the sum of P , Q and R ,

$$\therefore V = \frac{1}{3}h \times B + \frac{1}{3}h \times b + \frac{1}{3}h \sqrt{B \times b}$$

$$= \frac{1}{3}h(B + b + \sqrt{Bb}).$$

Q. E. D.

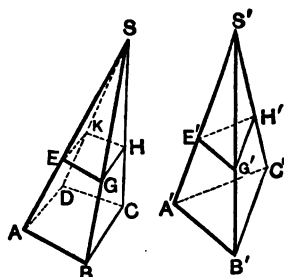
Proposition 20. Theorem.

636. *A frustum of any pyramid is equivalent to the sum of three pyramids, having the same altitude as the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the bases, of the frustum.*

Hyp. Let B denote the lower base, b the upper base, h the altitude, and V the volume of the frustum $ABCD-E$.

To prove $V = \frac{h}{3}(B + b + \sqrt{Bb})$.

Proof. Construct a triangular pyramid $S'-A'B'C'$ having the same altitude as $S-ABCD$, and its base $A'B'C'$ equivalent to $ABCD$, and in the same plane with it.



Then $\text{vol. } S-ABCD = \text{vol. } S'-A'B'C'.$ (633)

Produce the plane $EGHK$ of the upper base of the given frustum to cut the pyramid $S'-A'B'C'$ in $E'G'H'$.

Then $EGHK$ is equivalent to $E'G'H'.$ (626)

$\therefore \text{vol. } S-EGHK = \text{vol. } S'-E'G'H'.$ (633)

Taking the upper pyramids from the whole pyramids,

$\text{vol. } ABCD-EGHK = \text{vol. } A'B'C'-E'G'H'.$

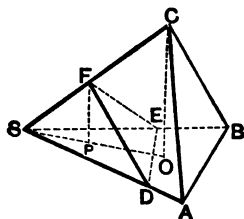
But, since $A'B'C'-E'$ is a frustum of a triangular pyramid whose lower and upper bases and altitude are B , b , and h , respectively,

$\therefore \text{vol. } A'B'C'-E' = \frac{1}{3}h(B + b + \sqrt{Bb}).$ (635)

$\therefore V = \frac{1}{3}h(B + b + \sqrt{Bb}).$ Q.E.D.

Proposition 21. Theorem.*

637. *Two tetraedrons which have a triedral angle of the one equal to a triedral angle of the other, are to each other as the products of the three edges of these triedral angles.*



Hyp. Let S-ABC, S-DEF be the two given tetraedrons, having the common triedral $\angle S$; and let V and V' denote their volumes.

To prove
$$\frac{V}{V'} = \frac{SA \times SB \times SC}{SD \times SE \times SF}.$$

Proof. Draw CO, FP $\perp$ to the plane SAB.

Then we may take the faces SAB, SDE as the bases, and CO, FP as the altitudes, of the triangular pyramids C-SAB, F-SDE.

$$\therefore \frac{V}{V'} = \frac{SAB \times CO}{SDE \times FP} = \frac{SAB}{SDE} \times \frac{CO}{FP}. \quad (633)$$

But
$$\frac{SAB}{SDE} = \frac{SA \times SB}{SD \times SE}. \quad (375)$$

And since the rt. $\triangle$ s SCO, SEP are similar,

$$\therefore \frac{CO}{FP} = \frac{SC}{SF}. \quad (307)$$

$$\therefore \frac{V}{V'} = \frac{SA \times SB \times SC}{SD \times SE \times SF}. \quad \text{Q.E.D.}$$

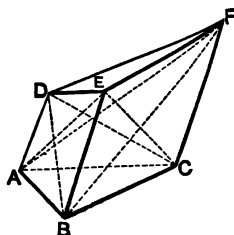
Proposition 22. Theorem.*

638. *A truncated triangular prism is equivalent to the sum of three pyramids whose common base is the lower base of the prism, and whose vertices are the three vertices of the upper base.*

Hyp. Let ABC-DEF be a truncated triangular prism.

To prove ABC-DEF equivalent to the sum of the three pyramids E-ABC, D-ABC, F-ABC.

Proof. Pass the planes AEC and DEC, dividing the truncated prism into the three pyramids E-ABC, E-ACD, and E-CDF.



The first pyramid E-ABC has the base ABC and the vertex E.

The second pyramid E-ACD and the pyramid B-ACD, have the same base ACD, and the same altitude, since their vertices E and B are in the line EB $\parallel$ to the base ACD.

$\therefore$ E-ACD is equivalent to B-ACD. (633)

But the pyramid B-ACD is the same as D-ABC.

$\therefore$ E-ACD is equivalent to D-ABC.

The third pyramid E-CDF and the pyramid B-ACF, have equivalent bases CDF and ACF (367), and the same altitude, since their vertices E and B are in the line EB $\parallel$ to CF, or to the plane ACFD.

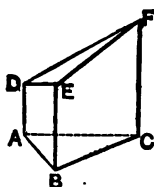
$\therefore$ E-CDF is equivalent to B-ACF. (633)

But the pyramid B-ACF is the same as F-ABC.

$\therefore$ E-CDF is equivalent to F-ABC.

$\therefore$ ABC-DEF is equivalent to the sum of the three pyramids E-ABC, D-ABC, F-ABC. Q.E.D.

639. COR. 1. *The volume of a truncated right triangular prism is equal to the product of its base by one-third the sum* of its lateral edges.*



For, the lateral edges AD, BE, CF, being $\perp$ to the base, are the altitudes of the three pyramids whose sum is equivalent to the truncated prism. (638)

$\therefore$ volume ABC-DEF

$$= \frac{1}{3}ABC \times AD + \frac{1}{3}ABC \times BE + \frac{1}{3}ABC \times CF \quad (632)$$

$$= ABC \times \frac{1}{3}(AD + BE + CF).$$

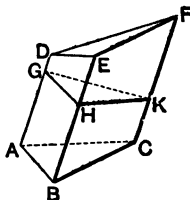
640. COR. 2. *The volume of any truncated triangular prism is equal to the product of its right section by one-third the sum of its lateral edges.*

For, the rt. section GHK divides the truncated triangular prism ABC-DEF into two truncated rt. prisms whose volumes are

$$GHK \times \frac{1}{3}(AG + BH + CK), \quad (639)$$

$$\text{and } GHK \times \frac{1}{3}(GD + HE + KF).$$

$\therefore$ their sum is $GHK \times \frac{1}{3}(AD + BE + CF).$



* Called the arithmetic mean of its lateral edges, also its mean height.

SIMILAR POLYEDRONS.

641. *Similar polyedrons* are those whose corresponding polyedral angles are equal, and which have the same number of faces, similar each to each, and similarly placed.

Homologous faces, edges, angles, etc., in similar polyedrons are faces, edges, angles, etc., which are similarly placed.

642. COR. 1. *The homologous edges of two similar polyedrons are proportional to each other.* (317)

643. COR. 2. *The homologous faces of two similar polyedrons are proportional to the squares of any two homologous edges.* (379)

644. COR. 3. *The entire surfaces of two similar polyedrons are proportional to the squares of any two homologous edges.* (296)

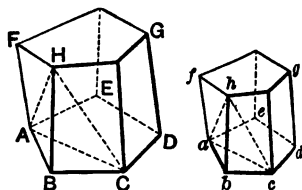
Proposition 23. Theorem.*

645. *Two similar polyedrons may be decomposed into the same number of tetraedrons, similar each to each, and similarly placed.*

Hyp. Let ABCDE-H and abcde-h be two similar polyedrons, H and h being homologous vertices.

To prove that they may be decomposed into the same number of tetraedrons, similar each to each, and similarly placed.

Proof. Decompose the polyedron AG into tetraedrons, by dividing all the faces not adjacent to H, into Δ s, and drawing st. lines from H to their vertices.



In the same manner, decompose the polyedron ag into tetraedrons having their common vertex at h homologous to H .

The two polyedrons are then decomposed into the same number of tetraedrons, similarly placed.

In the two homologous tetraedrons $H\text{-}ABC$ and $h\text{-}abc$, since the faces AD , HA , HC are similar respectively to the faces ad , ha , hc ,

(641)

$\therefore$ the Δ s ABC , HAB , HBC are similar respectively to the Δ s abc , hab , hbc .

(321)

$$\text{Since } \frac{AC}{ac} = \frac{AB}{ab}, \text{ and } \frac{AH}{ah} = \frac{AB}{ab}, \quad (307)$$

$$\therefore AC : ac = AH : ah.$$

$$\text{And since } \frac{AC}{ac} = \frac{BC}{bc}, \text{ and } \frac{HC}{hc} = \frac{BC}{bc}, \quad (307)$$

$$\therefore AC : ac = HC : hc.$$

$$\therefore \text{ the faces } HAC \text{ and } hac \text{ are similar.} \quad (313)$$

Also, the corresponding triedral $\angle$ s of these tetraedrons are equal.

(567)

$$\therefore \text{ the tetraedrons } H\text{-}ABC, h\text{-}abc \text{ are similar.} \quad (641)$$

In the same way it may be proved that any two homologous tetraedrons are similar.

$\therefore$ the two similar polyedrons may be decomposed into the same number of tetraedrons, similar each to each, and similarly placed.

Q.E.D.

646. COR. Any two homologous lines in two similar polyedrons are proportional to any two homologous edges.

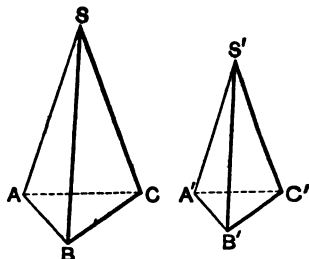
Proposition 24. Theorem.

647. *Two similar tetraedrons are to each other as the cubes of their homologous edges.*

Hyp. Let $SABC$, $S'A'B'C'$ be two similar tetraedrons; and let V and V' denote their volumes.

$$\text{To prove } \frac{V}{V'} = \frac{\overline{AB}^3}{\overline{A'B'}^3}.$$

Proof. Let the vertices S and S' be homologous.



Then, since the triedral $\angle$ s S and S' are equal, (641)

$$\therefore \frac{V}{V'} = \frac{SA \times SB \times SC}{S'A' \times S'B' \times S'C'} = \frac{SA}{S'A'} \times \frac{SB}{S'B'} \times \frac{SC}{S'C'}. \quad (637)$$

$$\text{But } \frac{SA}{S'A'} = \frac{SB}{S'B'} = \frac{SC}{S'C'} = \frac{AB}{A'B'}. \quad (641)$$

$$\therefore \frac{V}{V'} = \frac{AB}{A'B'} \times \frac{AB}{A'B'} \times \frac{AB}{A'B'} = \frac{\overline{AB}^3}{\overline{A'B'}^3}. \quad \text{Q.E.D.}$$

648. COR. 1. *Two similar polyedrons are to each other as the cubes of their homologous edges.*

For, two similar polyedrons may be decomposed into the same number of tetraedrons, similar each to each, and similarly placed (645); and any two homologous tetraedrons are to each other as the cubes of their homologous edges (647), or as the cubes of any two homologous edges of the polyedrons (646). Therefore the polyedrons themselves are to each other as the cubes of their homologous edges. (296)

649. COR. 2. *Similar prisms, or pyramids, are to each other as the cubes of their altitudes.*

650. COR. 3. *Similar polyedrons are to each other as the cubes of any two homologous lines.*

REGULAR POLYEDRONS.

651. A *regular polyedron* is one whose faces are all equal regular polygons, and whose polyedral angles are all equal.

Proposition 25. Theorem.*

652. *There can be only five regular convex polyedrons.*

Proof. At least three faces are necessary to form a polyedral angle, and the sum of its face-angles must be less than 360° . (566)

1. Because the angle of an equilateral triangle is 60° , each convex polyedral angle may have 3, 4, or 5 equilateral triangles. It cannot have 6 faces, because the sum of 6 such angles is 360° , reaching the limit. Therefore no more than three regular convex polyedrons can be formed with equilateral triangles; the *tetradron*, *octaedron*, and *icosaedron*.

2. Because the angle of a square is 90° , each convex polyedral angle may have 3 squares. It cannot have 4 squares because the sum of 4 such angles is 360° . Therefore only one regular convex polyedron can be formed with squares; the *hexaedron*, or *cube*.

3. Because the angle of a regular pentagon is 108° , each convex polyedral angle may have 3 regular pentagons. It cannot have 4 faces because the sum of 4 such angles is 432° . Therefore only one regular convex polyedron can be formed of regular pentagons; the *dodecaedron*.

Because the angle of a regular hexagon is 120° , and the angle of every regular polygon of more than 6 sides is yet greater than 120° , therefore there can be no regular convex polyedron formed of regular hexagons or of any regular polygons of more than 6 sides.

Therefore there can be only five regular convex polyedrons.

Proposition 26. Problem.*

653. *To construct the regular polyedrons, having given one of the edges.*

Given, the edge AB.

Required, to construct the regular polyedrons.

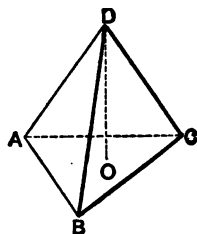
1. *The regular tetraedron.*

Cons. Upon AB construct the equilateral $\triangle ABC$.

At its centre O erect $OD \perp$ to the plane ABC so that $AD = AB$.

Join DA, DB, DC.

Then ABCD is a regular tetraedron.



Proof. Since the faces are each = the $\triangle ABC$, (496)

$\therefore$ the triedral $\angle$ s A, B, C, D are all equal. (567)

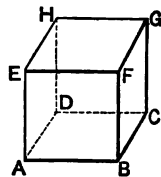
$\therefore$ ABCD is a regular tetraedron.

2. *The regular hexaedron, or cube.*

Cons. Upon AB construct the square ABCD.

Upon the sides of this square construct the four equal squares AF, BG, CH, DE, $\perp$ to the plane ABCD.

Then AG is a regular hexaedron.



Proof. Since the faces are equal squares, (Cons.)

$\therefore$ the triedral $\angle$ s A, B, C, D, E, F, G, H are all equal. (567)

$\therefore$ ABCD-E is a regular hexaedron, or cube.

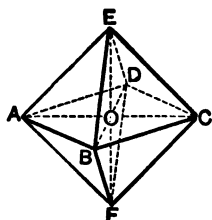
3. *The regular octaedron.*

Cons. Upon AB construct the square ABCD.

At its centre O erect EOF $\perp$ to the plane ABCD, so that $OE = OF = OA$.

Join the pts. E and F to all the vertices of the square.

Then E-ABCD-F is a regular octaedron.



Proof. The lines from E and F to A, B, C, D are all equal. (496)

And since the rt. $\triangle$ s AOE, AOB are equal, (104)

$$\therefore AE = AB.$$

$\therefore$ the twelve edges of the octaedron are all equal, and the faces are eight equal equilateral $\triangle$ s.

Since the diagonals BD and EF are equal and bisect each other at rt. $\angle$ s, (Cons.)

$\therefore$ BEDF is a square = ABCD.

And since AO is $\perp$ to BD and EF,

$\therefore$ AO is $\perp$ to the plane of BEDF. (500)

$\therefore$ pyramid A-BEDF = pyramid E-ABCD. (633)

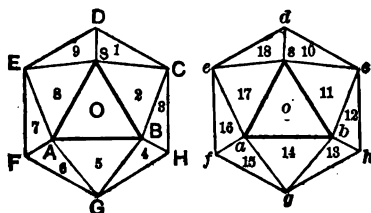
$\therefore$ polyedral $\angle$ A = polyedral $\angle$ E.

Similarly, it can be shown that any other two polyedral $\angle$ s are equal.

$\therefore$ E-ABCD-F is a regular octaedron.

5. *The regular icosaedron.*

Cons. Upon AB construct a regular pentagon ABCDE.



At its centre erect a $\perp$ to its plane, and in this $\perp$ take a pt. S so that $SA = AB$.

Join SA, SB, SC, SD, SE.

Then S-ABCDE is a regular pentagonal pyramid (496), and each of its faces is an equilateral Δ .

Complete the polyedral $\angle$ s at A and B by adding to each three equilateral Δ s each equal to SAB, and making the diedral $\angle$ s around A and B equal.

There is then formed a convex surface CDEF, etc., composed of ten equal equilateral Δ s.

Construct a second convex surface *cdef*, etc., equal to the first.

Then the two equal convex surfaces may be combined so as to form a single convex surface, which is the regular icosaedron.

Proof. Suppose the convexity of DG to be up, and the convexity of *dg* to be down.

Put the two surfaces together so that the pt. D, where two faces meet, falls upon the pt. *c*, where three faces meet. Then two consecutive face $\angle$ s of one surface will unite with three consecutive face $\angle$ s of the other. Thus, the two faces 1 and 9 will unite with the three faces 10, 11, 12, forming a polyedral $\angle$ of five faces equal to S, without in any way changing the form of either surface, since the

three dihedral $\angle$ s contained by 1 and 9, 10 and 11, 11 and 12 are those which belong to such a polyedral $\angle$.

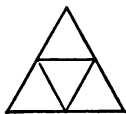
The vertex C will fall at h , and a like polyedral $\angle$ will be formed at that pt.; and so of all the others.

$\therefore$ the polyedral $\angle$ s are all equal, and the solid is a regular icosaedron. Q.E.F.

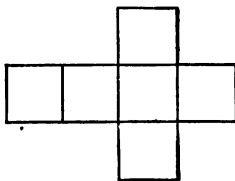
654. SCH. Models of the regular polyedrons may be easily constructed as follows:

Draw the following diagrams on cardboard, and cut them out. Then cut half way through the board in the dividing lines, and bring the edges together so as to form the respective polyedrons.

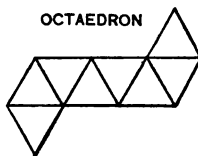
TETRAEDRON



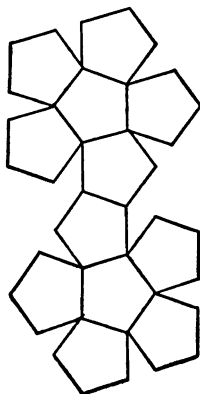
HEXAEDRON



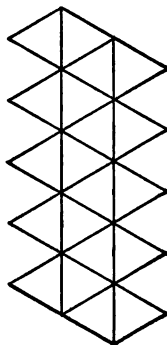
OCTAEDRON



DODECAEDRON



ICOSAEDRON



GENERAL PROPERTIES OF POLYEDRONS.

EULER'S THEOREM.†

Proposition 27. Theorem.*

655. *In any polyedron the number of edges increased by 2 is equal to the whole number of vertices and faces.‡*

Hyp. Let E , F , and V denote the number of edges, faces, and vertices respectively of the polyedron S -ABCD.

To prove $E + 2 = V + F$.

Proof. Beginning with one face ABCD, we have $E = V$.

If we annex to this a second face SAB, by applying one of its edges, as AB, to the corresponding edge of the first face, we form a surface having *one* edge AB, and *two* vertices A and B *common* to both faces; therefore, the whole number of edges is now one more than the whole number of vertices.

∴ for 2 faces ABCD and SAB, $E = V + 1$.

If we annex a third face SBC, adjacent to both ABCD and SAB, we form a new surface having *two* edges SB and BC, and *three* vertices S, B, C in common with the preceding surface; therefore the increase in the number of edges is again one more than the increase in the number of vertices.

∴ for 3 faces, $E = V + 2$.

In like manner, for 4 faces, $E = V + 3$.

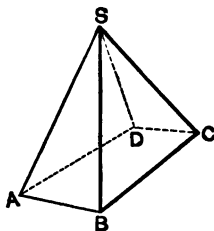
And so on until every face but one has been annexed.

∴ for $(F - 1)$ faces, $E = V + F - 2$.

But annexing the last face adds no edges nor vertices.

∴ for F faces, $E = V + F - 2$,

or $E + 2 = V + F$. Q.E.D.



† The proof of this theorem was first published by Euler (1752).

‡ This proof is due to Cauchy.

EXERCISES.

1. How many edges has the regular tetraedron?
2. How many edges has the regular hexaedron?
3. The frustum of a regular four-sided pyramid is 8 feet high, and the sides of its bases are 3 feet and 5 feet: find the height of an equivalent regular pyramid whose base is 10 feet square.

Proposition 28. Theorem.*

656. *The sum of the face-angles of any polyedron is equal to four right angles taken as many times as the polyedron has vertices less two.*

Hyp. Let E, F, and V denote the number of edges, faces, and vertices respectively, and S the sum of the face $\angle$ s of any polyedron.

To prove $S = 4 \text{ rt. } \angle \text{s } (V - 2).$

Proof. Since each edge is common to two faces,
 $\therefore$ the whole number of sides of the faces considered as independent polygons is $2E$.

If we form an exterior $\angle$ at each vertex of every polygon, the sum of the interior and exterior $\angle$ s at each vertex is $2 \text{ rt. } \angle \text{s}$.

And since the whole number of vertices in the faces is $2E$,

$\therefore$ the sum of all the interior and exterior $\angle$ s of the faces is $2 \text{ rt. } \angle \text{s} \times 2E$, or $4 \text{ rt. } \angle \text{s} \times E$.

Since the sum of the exterior $\angle$ s of each face is $4 \text{ rt. } \angle \text{s}$,
 $\angle \text{s}$, (151)

$\therefore$ the sum of the ext. $\angle$ s of the F faces is $4 \text{ rt. } \angle \text{s} \times F$.

$\therefore S + 4 \text{ rt. } \angle \text{s} \times F = 4 \text{ rt. } \angle \text{s} \times E$.

$\therefore S = 4 \text{ rt. } \angle \text{s } (E - F).$

But $E - F = V - 2.$ (655)

$\therefore S = 4 \text{ rt. } \angle \text{s } (V - 2).$ Q.E.D.

EXERCISES.

THEOREMS.

1. Show that a lateral edge of a right prism is equal to the altitude.

2. Show that the lateral faces of right prisms are rectangles.

3. Show that every section of a prism made by a plane parallel to the lateral edges is a parallelogram.

4. The lateral areas of right prisms of equal altitudes are as the perimeters of their bases.

5. The opposite faces of a parallelepiped are equal and parallel.

6. If the four diagonals of a quadrangular prism pass through a common point, the prism is a parallelepiped.

7. If any two non-parallel diagonal planes of a prism are perpendicular to the base, the prism is a right prism.

See (546).

8. Any straight line drawn through the centre of a parallelepiped, terminating in a pair of faces, is bisected at that point.

9. The volume of a triangular prism is equal to the product of the area of a lateral face by one-half the perpendicular distance of that face from the opposite edge.

10. In a cube the square of a diagonal is three times the square of an edge.

11. In any parallelepiped, the sum of the squares of the twelve edges is equal to the sum of the squares of its four diagonals.

12. In any quadrangular prism, the sum of the squares of the twelve edges is equal to the sum of the squares of its four diagonals, plus eight times the square of the line joining the common middle points of the diagonals taken two and two.

13. A plane passing through a triangular pyramid,

parallel to one side of the base and to the opposite lateral edge, intersects its faces in a parallelogram.

14. The lateral surface of a pyramid is greater than the base.

Project the vertex on the base, etc.

15. The four middle points of two pairs of opposite edges of a tetraedron are in one plane, and at the vertices of a parallelogram.

(617).

16. The three lines joining the middle points of the three pairs of opposite edges of a tetraedron intersect in a point which bisects them all.

17. In a tetraedron, the planes passed through the three lateral edges and the middle points of the edges of the base intersect in a straight line.

The intersections of the planes with the base are medials of the base; $\therefore$ etc.

18. The lines joining each vertex of a tetraedron with the point of intersection of the medial lines of the opposite face all meet in a point, which divides each line in the ratio 1 : 4.

NOTE.—This point is the *centre of gravity* of the tetraedron.

19. The plane which bisects a diedral angle of a tetraedron divides the opposite edge into segments which are proportional to the areas of the adjacent faces.

See (303), (359).

20. The volume of a truncated triangular prism is equal to the product of the area of its lower base by the perpendicular upon the lower base let fall from the intersection of the medial lines of the upper base.

21. The volume of a truncated parallelopiped is equal to the product of the area of its lower base by the perpendicular from the centre of the upper base upon the lower base.

22. The volume of a truncated parallelopiped is equal to the product of a right section by one-fourth the sum of its four lateral edges.

See (640).

23. Any plane passed through the centre of a parallelopiped divides it into two equivalent solids (640).

24. The portion of a tetraedron cut off by a plane parallel to one of its faces is a tetraedron similar to the whole tetraedron.

25. When two tetraedrons have a diedral angle of the one equal to a diedral angle of the other, and the faces including these angles similar each to each, and similarly placed, the tetraedrons are similar.

26. Two polyedrons composed of the same number of tetraedrons, similar each to each, and similarly placed, are similar.

NUMERICAL EXERCISES.

27. Find the lateral area of a right prism whose altitude is 14 inches and perimeter of the base 16 inches.

Ans. 224 square inches.

28. Find the volume of a prism the area of whose base is 24 square inches and altitude 7 feet. *Ans.* 168 cubic feet.

29. Find the surface of a cube whose sides are each 11 inches.

Ans. 726 square inches.

30. Find the surface of a cubical cistern whose contents are 373,248 cubic inches.

Ans. 180 square feet.

31. Find the lateral surface of a right prism whose altitude is 2 feet, and whose base is a regular hexagon of which each side is 10 inches long.

32. Find the depth of a cubical cistern which shall hold 1600 gallons, each gallon being 231 cubic inches.

Ans. 5.98 feet.

33. If the dimensions of a rectangular parallelopiped are 20.5 feet, $18\frac{1}{2}$ feet, and 6.75 feet, what is the edge of an equivalent cube?

Ans. 11.4 feet.

34. Find the depth of a cubical box which shall contain 100 bushels of grain, each bushel holding 2150.42 cubic inches.

Ans. 4.9 feet.

35. If the dimensions of a rectangular parallelopiped are 9, 16, and 25, what is the edge of an equivalent cube?

36. Find the dimensions of the base of a rectangular

parallelopiped whose volume is 60, surface 94, and altitude 3.

37. Find the ratio of two rectangular parallelopipeds, if their altitudes are each 4 feet, and their bases 6 feet by 3 feet, and 12 feet by 8 feet, respectively.

38. Find the ratio of two rectangular parallelopipeds, if their dimensions are 5, 6, 8, and 10, 12, 16, respectively.

39. Find the volume of a right triangular prism, if the height is 8 inches, and the sides of the base are 6, 5, and 5 inches.

40. Find the lateral area of a right pyramid whose slant height is 4 feet, and whose base is a regular octagon of which each side is 3 feet long.

41. Find the volume of a right quadrangular pyramid whose altitude is 12, and whose base is 4 feet square.

42. Find the volume of a pyramid whose altitude is 20 feet, and whose base is a rectangle 8 feet by 7.

43. Find the lateral area of a right pentagonal pyramid whose slant height is 18 inches, and each side of the base 6 inches.

Ans. 270 square inches.

44. Find the volume of a pyramid whose altitude is 20 inches, and whose base is a regular hexagon, each side being 6 inches.

Ans. 623.5386 cubic inches.

45. Find the lateral area and volume of a right triangular pyramid, the sides of whose bases are 3, 5, and 6, and whose altitude is 6.

See (398), Ex. 2.

46. Find the volume of the frustum of a square pyramid, the sides of whose bases are 8 and 6 feet, and whose altitude is 12 feet.

Ans. 592 square feet.

47. If a plane be passed parallel to the base of the pyramid in Ex. 44, midway between the vertex and base, find the lateral area and volume of the frustum.

48. The slant height of the frustum of a right pyramid is 6 feet, and the perimeters of the two bases are 18 feet and 12 feet: what is the lateral area of the frustum?

49. Find the slant height of the pyramid whose frustum is given in Ex. 48.

50. Find the volume of the frustum of a regular triangular pyramid, the sides of whose bases are 8 and 6, and whose lateral edge is 5.

51. A pyramid 18 feet high has a base containing 169 square feet. How far from the vertex must a plane be passed parallel to the base so that the section may contain 81 square feet?

52. The base of a pyramid contains 169 square feet; a plane parallel to the base and 5 feet from the vertex cuts a section containing 81 square feet: find the height of the pyramid.

53. A pyramid 16 feet high has a square base 10 feet on a side. Find the area of a section made by a plane parallel to the base and 6 feet from the vertex.

54. The base of a regular pyramid is a hexagon of which the side is 4 feet. Find the height of the pyramid if the lateral area is eight times the area of the base.

55. Find the total surface of a regular pyramid, (1) when each side of its square base is 12 feet, and the slant height is 24 feet; (2) when each side of its square base is 30 feet, and the perpendicular height is 94 feet; and (3) when each side of its triangular base is 8 feet, and the slant height is 24 feet.

56. Find the volume of a regular pyramid when each side of its square base is 80 feet, and the lateral edge is 202 feet.

57. Find the volume of a regular pyramid whose base is an equilateral triangle inscribed in a circle of radius 40 feet, and whose slant height is 48 feet.

58. Find the lateral edge, lateral area, and volume, (1) of a regular hexagonal pyramid, each side of whose base is 2, and whose altitude is 12; (2) of a frustum of a regular triangular pyramid, the sides of whose bases are $5\sqrt{3}$ and $\sqrt{3}$, and whose altitude is 3; (3) of a frustum of a quadrangular pyramid, the sides of whose bases are 11 and 1, and whose

altitude is 12; and (4) of a frustum of a regular hexagonal pyramid, the sides of whose bases are 12 and 4, and whose altitude is 12.

59. Find the volume of the frustum of a regular square pyramid, the sides of whose bases are 40 and 16 feet, and whose slant height is 20 feet.

60. The volume of a frustum of a regular hexagonal pyramid is 12 cubic feet, the sides of the bases are 2 and 1 feet: find the height of the frustum.

61. Find the difference between the volume of the frustum of a regular quadrangular pyramid, the sides of whose bases are 4 and 3 feet, and the volume of a prism of the same altitude, whose base is a section of the frustum parallel to its bases and midway between them.

62. Find the dimensions of a cube whose surface shall be numerically equal to its contents.

63. A regular pyramid 8 feet high is transformed into a regular prism with an equivalent base: find the height of the prism.

64. A cube whose edge is 3 feet is transformed into a right prism whose base is a rectangle 3 feet by 2 feet: find the height of the prism.

65. The height of the frustum of a regular quadrangular pyramid is 12 feet, and the sides of its bases are 6 and 10 feet: find the height of an equivalent regular pyramid whose base is 24 feet square.

66. A mound of earth is raised with plane sloping sides and rectangular bases; the dimensions at the bottom are 80 yards by 10, at the top 70 yards by 1, and it is 5 yards high: find its cubical contents.

67. A bath 6 feet deep is excavated; the area of the surface at the top is 100 square yards, at the bottom 81 square yards: find the number of gallons of water it will hold.

68. A railway embankment across a valley has the following measures: width at top 20 feet, at base 45 feet,

height 11 feet, length at top 1020 yards, at base 960 yards: find its volume.

69. The altitude of a prism is 9 feet and the perimeter of the base 6 feet: find the altitude and perimeter of the base of a similar prism one-third as great.

70. A pyramid is cut by a plane parallel to the base and midway between the vertex and base: find the ratio of the volume cut off to the whole volume.

71. The length of one of the lateral edges of a pyramid is 16 inches. How far from the vertex will this edge be cut by a plane parallel to the base, which divides the pyramid into two equivalent parts?

72. The height of the frustum of a pyramid is 16 inches, and two homologous edges of its bases are 8 and 6 inches: find the ratio of the volume of the frustum to that of the entire pyramid.

73. Show that the volume of a regular tetraedron is equal to the cube of its edge multiplied by $\frac{1}{12} \sqrt{2}$.

74. Show that the volume of a regular octaedron is equal to the cube of its edge multiplied by $\frac{1}{3} \sqrt{2}$.

75. Find the volume of a regular icosaedron whose edges are each 20 feet.

76. Verify Euler's Theorem in the case of all the regular polyedrons.

77. Find the number of edges in a pyramid, and in a prism, on a polygon of n sides as base.

PROBLEMS.

78. Given three indefinite straight lines in space which do not intersect, to construct a parallelopiped which shall have three of its edges on these lines.

79. To cut a solid tetraedral angle by a plane, so that the section shall be a parallelogram.

Book VIII.*

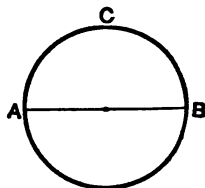
THE SPHERE.†

CIRCLES OF THE SPHERE AND TANGENT PLANES.

DEFINITIONS.

657. A *sphere* is a solid bounded by a curved surface, every point of which is equally distant from a point within it, called the *centre*.

A sphere may be generated by the revolution of a semicircle ACB about its diameter AB as an axis.



658. A *radius* of a sphere is a straight line drawn from the centre to any point of the surface.

A *diameter* of a sphere is a straight line passing through its centre, and terminating at both ends in the surface.

All the radii of a sphere are equal; and all the diameters are equal, since each is the sum of two radii.

659. A line or plane is *tangent* to a sphere when it has but one point in common with the surface of the sphere.

660. Two spheres are *tangent* to each other when their surfaces have but one point in common.

661. The common point is called the *point of contact*, or *point of tangency*.

662. Two spheres are *concentric* when they have the same centre.

663. Two spheres are *equal* when they have equal radii.

* This book treats of the properties and relations of those parts of the surface of a sphere which are bounded by arcs of great circles.

† In teaching this subject, as well as in teaching *Spherical Trigonometry*, the class-room should be furnished with a spherical black-board, on which the student should draw the diagrams of spherical surfaces.

Proposition 1. Theorem.

664. Every section of a sphere made by a plane is a circle.

Hyp. Let ACB be a plane section of the sphere whose centre is O.

To prove that ACB is a $\odot$.

Proof. Draw $OO' \perp$ to the plane ACB meeting it in O' , and draw OA, OC to any two pts. in the section.

Since OA and OC are radii of the sphere,

$$\therefore OA = OC. \quad (658)$$

And since OA and OC are equal oblique lines from O to the plane ACB,

$$\therefore O'A = O'C. \quad (498)$$

But A and C are any two pts. in the section ACB.

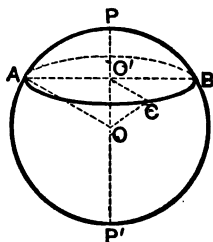
$\therefore$ the curve ACB is a $\odot$ whose centre is O' . Q.E.D.

665. DEF. The circular section of a sphere by a plane is called a *circle of the sphere*.

If the plane passes through the centre of the sphere, the section is a circle whose radius is equal to the radius of the sphere. This circle is called a *great circle of the sphere*.

A section made by a plane which does not pass through the centre of the sphere is called a *small circle*.

666. DEF. The two points in which a diameter of the sphere, perpendicular to the plane of a circle, meets the surface of the sphere, are called the *poles* of the circle, and this diameter is called the *axis* of the circle; thus, P, P' are the poles of the circle ACB, and PP' is its axis.*



* P' is said to be antipodal to P.

667. COR. 1. *The line through the centre of a circle of the sphere, perpendicular to its plane, passes through the centre of the sphere.* (664)

Therefore, *the axis of a circle passes through its centre* (666), *and all parallel circles have the same axis and the same poles.*

668. COR. 2. *All great circles of a sphere are equal.* (665)

669. COR. 3. *All small circles at equal distances from the centre of the sphere are equal; and of two circles unequally distant from the centre, the nearer is the larger, and conversely.*

670. COR. 4. *Every great circle bisects the sphere and its surface.*

For, if the two parts are separated, and then placed on the common base, with their convexities turned the same way, their surfaces will coincide, since all the points of either are equally distant from the centre. (657)

671. COR. 5. *Any two great circles bisect each other.*

For, the common intersection of their planes passes through the centre of the sphere, and is a diameter of each circle.

672. COR. 6. *An arc of a great circle may be drawn through any two given points on the surface of a sphere.*

For, the two given pts. with the centre of the sphere determine the plane of a great $\odot$ passing through the two given pts. (484)

If the two pts. are the extremities of a diameter, the position of the circle is not determined: for the two given pts. and the centre being in the same st. line, an indefinite number of planes can be drawn through them. (482)

673. COR. 7. *An arc of a circle may be drawn through any three given points on the surface of a sphere.*

For, the three pts. determine a plane. (484)

NOTE.—By the distance between two points on a sphere is meant the arc of a great circle joining them.

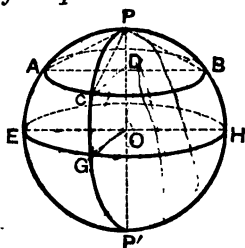
Proposition 2. Theorem.

674. *All points in the circumference of a circle of a sphere are equally distant from each of its poles.*

Hyp. Let P, P' be the poles of the $\odot ACB$, where A, C, B are any pts. on its $\odot ce$.

To prove arcs PA, PC, PB equal.

Proof. Since PP' is a line through the centre of the $\odot ACB \perp$ to its plane,



$\therefore$ the st. lines PA, PC, PB are equal. (496)

$\therefore$ the arcs PA, PC, PB are equal. (198)

Similarly, the arcs $P'A, P'C, P'B$ are equal. Q.E.D.

675. *DEF.* The distance from any point in the circumference of a circle to its nearest pole is called *the polar distance* of the circle.

676. *COR. 1.* *The polar distance of a great circle is a quadrant.* Thus, PE, PG are quadrants, for they are the measures of the rt. $\angle$ s POE, POG , whose vertices are at the centre of the sphere. (237)

677. *COR. 2.* *If a point P on the surface of a sphere is at a quadrant's distance from the two points E, G, of an arc of a great circle, it is the pole of that arc.*

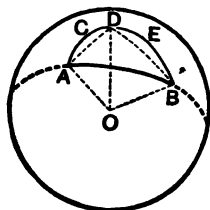
For, the $\angle$ s POE, POG are rt. $\angle$ s; $\therefore$ the radius OP is $\perp$ to the plane of the arc EG (500); and $\therefore P$ is the pole of the arc EG . (666)

678. *SCH.* In Spherical Geometry, the term *quadrant* usually means a quadrant of a *great circle*.

Proposition 3. Theorem.

679. *The shortest distance on the surface of a sphere, between any two points on the surface, is the arc of a great circle, not greater than a semi-circumference, joining the two points.*

Hyp. Let AB be an arc of a great $\odot$, not greater than a semi- $\odot$, joining any two pts. A and B on the sphere; and let $ACEB$ be any other line in the surface joining A and B .



To prove arc $AB < ACEB$.

Proof. Take any pt. D in $ACEB$, pass arcs of great $\odot$ s through A, D and B, D ; and join O , the centre of the sphere, with A, D, B .

Then, since the $\angle$ s AOB, AOD, DOB are the face $\angle$ s of the trihedral $\angle$ whose vertex is at O ,

$$\therefore \angle AOD + \angle DOB > \angle AOB. \quad (565)$$

But the $\angle$ s AOD, DOB, AOB are measured by the arcs AD, DB, AB , respectively. (236)

$$\therefore \text{arc } AD + \text{arc } DB > \text{arc } AB.$$

Similarly, joining any pt. in ACD with A and D by arcs of great $\odot$ s, the sum of the arcs is greater than arc AD ; and joining any pt. in DEB with D and B by arcs of great $\odot$ s, the sum is greater than arc DB .

If this process is indefinitely repeated, the line from A to B , on the arcs of the great $\odot$ s, will continually increase and approach the line $ACEB$; that is, the sum of the arcs of the great $\odot$ s will approach $ACEB$ as the limit, and will always be greater than AB .

$$\therefore \text{arc } AB < ACEB.$$

Q.E.D.

Proposition 4. Theorem.

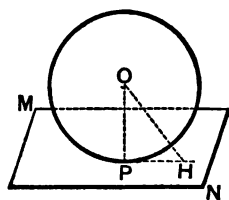
680. *A plane perpendicular to a radius of a sphere at its extremity is tangent to the sphere.*

Hyp. Let the plane MN be $\perp$ to the radius OP at its extremity P .

To prove MN tangent to the sphere.

Proof. Take any other pt. H in the plane, and join OH .

Because the $\perp$ is the shortest distance from a point to a plane,



(497)

$$\therefore OP < OH.$$

$\therefore$ the pt. H is without the sphere.

But H is any pt. of MN except P .

$\therefore$ every pt. of MN except P is without the sphere.

$\therefore$ plane MN is tangent to the sphere at the pt. P . (659)

Q.E.D.

681. COR. 1. *Every straight line perpendicular to a radius at its extremity is tangent to the sphere.* (659)

682. COR. 2. *Every plane or line tangent to a sphere is perpendicular to the radius drawn to the point of contact.*

683. COR. 3. *A straight line tangent to any circle of a sphere lies in the plane tangent to the sphere at the point of contact.*

684. SCH. 1. Any straight line drawn in a tangent plane through the point of contact is tangent to the sphere at that point.

685. SCH. 2. Any two straight lines, tangent to the sphere at the same point, determine the tangent plane at that point.

Proposition 5. Theorem.*

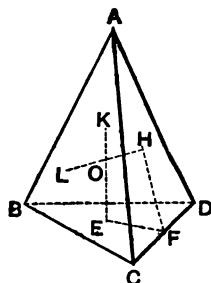
686. *Through any four points not in the same plane, one spherical surface, and only one, may pass.*

Hyp. Let A, B, C, D be the four pts. not in the same plane.

To prove that one spherical surface, and no more, may pass through A, B, C, D.

Proof. Let E, H be the centres of the $\odot$ s circumscribed about the $\triangle$ s BCD, ACD, respectively.

Draw $EK \perp$ to plane BCD, and $HL \perp$ to plane ACD.



Every pt. in EK is equally distant from the pts. B, C, D, and every pt. in HL is equally distant from the pts. A, C, D. (496)

Join E and H to F, the middle pt. of CD.

EF and HF are each $\perp$ to CD. (203)

$\therefore$ the plane through EF and HF is $\perp$ to CD, (500)

and $\therefore$ this plane is $\perp$ to both planes BCD, ACD. (540)

Since HL is $\perp$ to the plane ACD at H, (Cons.)

$\therefore$ HL lies in the plane EFH. (538)

Similarly, it may be shown that EK lies in this plane.

$\therefore$ the $\perp$ s EK, HL lie in the same plane; and, being $\perp$ to planes which are not $\parallel$, cannot be $\parallel$, and $\therefore$ must meet at some pt. O.

Since O is in the $\perp$ s EK and HL, it is equally distant from B, C, D, and from A, C, D. (496)

$\therefore$ O is equally distant from A, B, C, D; and the sphere described with O as a centre and OA as a radius, will pass through the pts. A, B, C, D.

Also, since the centre of any sphere passing through the four pts. A, B, C, D, must be in the $\perp$ s EK, HL, (498)

$\therefore$ the intersection O is the centre of the only sphere that can pass through the four given pts. Q.E.D.

687. COR. 1. *A sphere may be circumscribed about any tetraedron.*

688. COR. 2. *The four perpendiculars to the faces of a tetraedron through their centres meet at the same point.*

689. COR. 3. *The six planes which bisect at right angles the six edges of a tetraedron all intersect in the same point.*

Proposition 6. Theorem.*

690. *A sphere may be inscribed in a given tetraedron.*

Hyp. Let ABCD be the given tetraedron.

To prove that a sphere may be inscribed in ABCD.

Proof. Bisect any three of the dihedral $\angle$ s which have one face common, as BC, CD, BD, by the planes OBC, OCD, OBD, respectively.

Since O is in the bisector of the dihedral $\angle$ BC, it is equally distant from the faces ABC and BCD. (549)

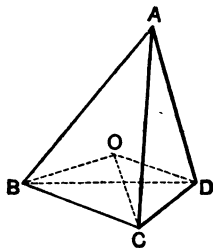
In the same way O is equally distant from the faces ACD and BCD, and from BAD and BCD.

$\therefore$ the pt. O is equally distant from the four faces of the tetraedron.

$\therefore$ a sphere described with O as a centre, and with a radius equal to the common distance of O from any face, will be tangent to each face, and will be inscribed in the tetraedron. (659)

Q.E.D.

691. COR. *The six planes which bisect the six dihedral angles of a tetraedron intersect in a point.*



SPHERICAL TRIANGLES AND POLYGONS.

DEFINITIONS.

692. The angle between two intersecting curves is the angle included between their tangents at the point of intersection.

When the two curves are arcs of great circles the angle is called a *spherical angle*.

693. A *spherical polygon* is a portion of the surface of a sphere bounded by three or more arcs of great circles.

The bounding arcs are the *sides* of the polygon; the points of intersection of the sides are the *vertices* of the polygon; and the angles which the sides make with each other are the *angles* of the polygon.

A *diagonal* of a spherical polygon is an arc of a great circle joining any two vertices which are not consecutive.

694. A *spherical triangle* is a spherical polygon of three sides.

A spherical triangle is *right* or *oblique*, *scalene*, *isosceles*, or *equilateral*, in the same cases as a plane triangle.

NOTE.—Between any two points two arcs of great circles may be drawn, the one less, and the other greater, than a semi-circumference. In the present treatise the arcs less than a semi-circumference will be taken, unless otherwise stated.

695. A *spherical pyramid* is a portion of the sphere bounded by a spherical polygon and the planes of the sides of the polygon. The centre of the sphere is the *vertex* of the pyramid, and the spherical polygon is its *base*.

696. Two spherical polygons are *equal* if they can be applied one to the other, so as to coincide.

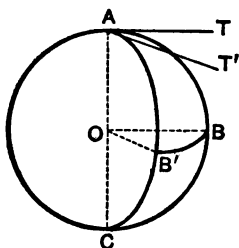
697. Since the sides of a spherical polygon are arcs, they are usually expressed in *degrees*, *minutes*, and *seconds*.

NOTE.—Because the apparent position of the heavenly bodies is referred to an imaginary spherical surface whose centre we occupy, the geometry of the surface of the sphere early attracted attention. It cannot be studied to any great extent without a knowledge of Trigonometry, but a few important propositions may be given, which illustrate this branch of Geometry.

Proposition 7. Theorem.

698. *A spherical angle is measured by the arc of a great circle described with its vertex as a pole and included between its sides, produced if necessary.*

Hyp. Let ABC , $AB'C$ be two arcs of great $\odot$ s intersecting at A ; let AT , AT' be the tangents to these arcs at A ; and let OBB' be a plane through the centre O $\perp$ to AC , intersecting the sphere in the great $\odot$ BB' .



To prove that the spherical $\angle BAB'$ is measured by the arc BB' .

Proof. Since TA and $T'A$ are respectively in the planes of the arcs BA and $B'A$, and are $\perp$ to their intersection AC ,
(210)

$$\therefore \angle TAT' = \text{the diedral } \angle BACB'.$$

$$\text{Also, } \angle BOB' = \text{the diedral } \angle BACB', \quad (528)$$

$$\text{and it is measured by the arc } BB'. \quad (236)$$

$$\text{But the spherical } \angle BAB' \text{ is measured by } \angle TAT'. \quad (692)$$

$$\therefore \text{ it is measured by the arc } BB'. \quad \text{Q.E.D.}$$

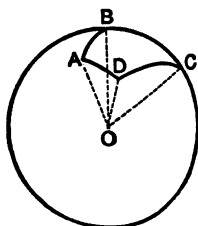
699. COR. 1. *A spherical angle is equal to the diedral angle between the planes of the two circles.*

700. COR. 2. *If two arcs of great circles cut each other, their vertical angles are equal.*

701. COR. 3. *The angles of a spherical triangle are equal to the diedral angles between the planes of the sides of the triangle.*

RELATION OF A SPHERICAL POLYGON TO A POLYEDRAL ANGLE.

702. Because the planes of all great circles pass through the centre of the sphere, therefore the planes of the sides of a spherical polygon form a polyedral angle at the centre O whose face-angles AOB , BOC , etc., are measured by the sides AB , BC , etc., of the polygon (236), and whose diedral angles OA , OB , etc., are equal to the angles A , B , etc., of the spherical polygon ABC , etc. (699).



We may therefore speak of *all* the parts of a spherical polygon as *angles*, meaning thereby the face-angles, and the diedral angles between the faces, of the polyedral angle whose vertex is the centre of the sphere, and base the spherical polygon.

It follows, therefore, that the properties of a spherical polygon and a polyedral angle are mutually convertible. Hence:

703. *From any relation proved between the face, and diedral, angles of a polyedral angle, we may infer the same relation between the sides and angles of a spherical polygon.*

And conversely: *From any relation proved between the sides and angles of a spherical polygon, we may infer the same relation between the face, and diedral, angles of a polyedral angle.* Therefore:

704. *Each side of a spherical triangle is less than the sum of the other two sides (565).*

705. *The sum of the sides of a spherical polygon is less than a circumference (566).*

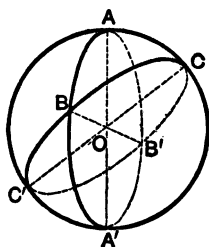
706. *Two mutually equilateral triangles on the same, or on equal spheres, are mutually equiangular, and are either equal or symmetrical (567).*

SYMMETRICAL SPHERICAL TRIANGLES.

707. *Symmetrical Spherical Triangles* are those in which the sides and angles of the one are equal respectively to the sides and angles of the other, but arranged in the reverse order.

Thus, the spherical triangles ABC and $A'B'C'$ are symmetrical when the vertices of the one are at the ends of the diameters from the vertices of the other.*

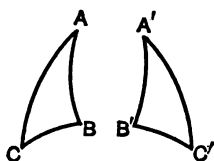
The corresponding triedral angles $O-ABC$ and $O-A'B'C'$ are also symmetrical. (567)



In the same way we may form two symmetrical polygons of any number of sides.

Two symmetrical triangles are mutually equilateral and equiangular; yet in general they cannot be made to coincide.

Thus, if in the symmetrical triangles ABC , $A'B'C'$, AB is made to coincide with $A'B'$ to bring the vertex C upon the corresponding vertex C' , the two convex surfaces would have to be brought together. The triangles are



in fact right-handed and left-handed, and, though corresponding and equal in every detail, can no more be conceived as superposed on one another so as to occupy the same space, than the form of the right hand on that of the left hand.

* Antipodal.

Proposition 8. Theorem.

708. *Two symmetrical spherical triangles are equivalent.*

Hyp. Let ABC , $A'B'C'$ be two symmetrical spherical Δ s with their homologous vertices diametrically opposite each other. (707)

To prove $\text{area } ABC = \text{area } A'B'C'$.

CASE I. *When the triangles are isosceles.*

Proof. Let $BA = BC$, and $B'A' = B'C'$.

If $\angle B$ be placed on the equal $\angle B'$ (702), the convexities of the sphere being on the same side, the side BA will fall on $B'C'$, and BC on $B'A'$.

And since $BA = B'C'$, and $BC = B'A'$,

$\therefore A$ will fall on C' , and C on A' .

$\therefore$ the two Δ s coincide throughout and are identically equal.

CASE II. *When the triangles are not isosceles.*

Proof. Let P and P' be the poles of the small $\odot$ s passing through the pts. A, B, C , and A', B', C' , respectively.*

Draw the great $\odot$ arcs PA, PB, PC , and $P'A', P'B', P'C'$.

Then $PA = PB = PC$. (674)

Similarly, $P'A' = P'B' = P'C'$.

Since two symmetric Δ s are mutually equilateral, (707)

$\therefore P'A' = PA, P'B' = PB, P'C' = PC$.

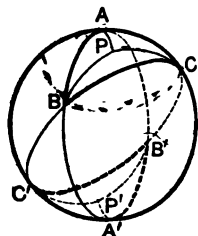
$\therefore$ the Δ s PAC and $P'A'C'$, PCB and $P'C'B'$, PBA and $P'B'A'$ are respectively isosceles symmetric Δ s.

$\therefore$ they are identically equal. (Case I)

Because sum of Δ s $PAC, PCB, PBA = \text{area } ABC$;
and sum of Δ s $P'A'C', P'C'B', P'B'A' = \text{area } A'B'C'$,

$\therefore \text{area } ABC = \text{area } A'B'C'$.

Q. E. D.



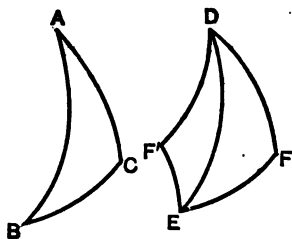
* The circle which passes through the three points A, B, C can only be a small circle of the sphere; for if it were a great circle, the three sides AB, BC, CA , would lie in one plane, and the triangle ABC would be reduced to one of its sides.

Proposition 9. Theorem.*

709. *Two triangles on the same, or on equal spheres, having two sides and the included angle of one equal respectively to two sides and the included angle of the other, are either equal or equivalent.*

Hyp. Let ABC , DEF be two Δ s having the side $AB = DE$, the side $AC = DF$, and the $\angle A = \angle D$.

CASE I. *When the given parts of the two Δ s are arranged in the same order.*



To prove $\Delta ABC = \Delta DEF$.

Proof. The ΔABC may be placed on the ΔDEF , as in the corresponding case of plane Δ s, and will coincide with it. (104)

CASE II. *When the given parts are arranged in inverse order, as in Δ s ABC , DEF' .*

To prove Δ s ABC and DEF' equivalent.

Proof. Let the ΔDEF be symmetrical with the $\Delta DEF'$, having its sides and $\angle$ s equal respectively to those of DEF' . Then in the Δ s ABC , DEF , we have

$$AB = DE, AC = DF, \angle A = \angle D,$$

and the parts arranged in the same order.

$$\therefore \Delta ABC = \Delta DEF. \quad (\text{Case I})$$

$$\text{But} \quad \Delta DEF \text{ is equivalent to } \Delta DEF'. \quad (708)$$

$$\therefore \Delta ABC \text{ is equivalent to } \Delta DEF'. \quad \text{Q.E.D.}$$

Proposition 10. Theorem.*

710. *Two triangles on the same, or on equal spheres, having a side and the two adjacent angles of one equal respectively to a side and the two adjacent angles of the other, are either equal or equivalent.*

Proof. One of the Δ s, or the Δ symmetric with it, may be applied to the other, as in the corresponding case of plane Δ s.

(105)

Q.E.D.

Proposition 11. Theorem.*

711. *Two mutually equilateral triangles, on the same, or on equal spheres, are either equal or equivalent.*

Proof. They are mutually equiangular, and equal or symmetrical.

(706)

$\therefore$ they are either equal or equivalent.

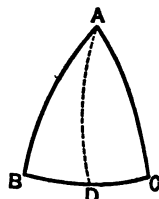
(708)

Q.E.D.

712. COR. 1. *In an isosceles spherical triangle, the angles opposite the equal sides are equal.*

For, in the ΔABC , let $AB = AC$; pass the arc AD of a great $\odot$ through A and the mid. pt. of BC ; then the Δ s ABD and ACD are mutually equilateral, and $\therefore$ mutually equiangular.

(706)

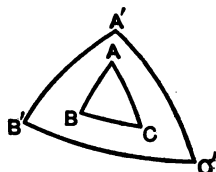


713. COR. 2. *The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the middle of the base is perpendicular to the base, and bisects the vertical angle.*

POLAR TRIANGLES.

714. One spherical triangle is called the *polar triangle* of a second spherical triangle when the sides of the first triangle have their poles at the vertices of the second.

Thus, if A, B, C are the poles of the arcs of the great circles $B'C', C'A', A'B'$, respectively, then $A'B'C'$ is the polar triangle of ABC .



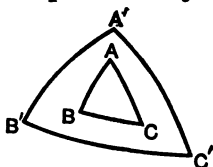
The great circles, of which $B'C', C'A', A'B'$ are arcs, will form three other spherical triangles on the hemisphere. But the one which is the polar of ABC is that whose vertex A' , homologous to A , lies on the same side of BC as the vertex A ; and in the same way with the other vertices.

Proposition 12. Theorem.

715. *If the first of two spherical triangles is the polar triangle of the second, then the second is the polar triangle of the first.*

Hyp. Let $A'B'C'$ be the polar $\triangle$ of ABC .

To prove that ABC is the polar $\triangle$ of $A'B'C'$.



Proof. Because B is the pole of $A'C'$, (714)

$\therefore BA'$ is a quadrant. (676)

Because C is the pole of $A'B'$,

$\therefore CA'$ is a quadrant.

$\therefore A'$ is the pole of BC . (677)

In like manner, B' is the pole of AC , and C' the pole of AB .

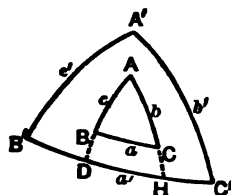
Also, A and A' are on the same side of $B'C'$, and so of the other vertices.

$\therefore ABC$ is the polar $\triangle$ of $A'B'C'$. Q.E.D.

Proposition 13. Theorem.

716. *In two polar triangles, each angle of one is the supplement of the side opposite to it in the other.*

Hyp. Let ABC , $A'B'C'$ be a pair of polar Δ s in which A , B , C , and A' , B' , C' denote the $\angle$ s, and a , b , c , and a' , b' , c' denote the sides.



To prove

$$\begin{aligned} A &= 180^\circ - a', \\ B &= 180^\circ - b', \\ C &= 180^\circ - c', \\ A' &= 180^\circ - a, \\ B' &= 180^\circ - b, \\ C' &= 180^\circ - c. \end{aligned}$$

Proof. Produce the sides AB , AC , until they meet $B'C'$ at D and H .

Because A is the pole of $B'C'$, AD and AH are quadrants. (676)

$\therefore \angle A = \text{arc } DH.$ (698)

Because B' is the pole of AH , arc $B'H = 90^\circ$.

Because C' is the pole of AD , arc $C'D = 90^\circ$.

$\therefore \text{arc } B'H + \text{arc } C'D = 180^\circ.$

But $\text{arc } B'H + \text{arc } C'D = \text{arc } B'C' + \text{arc } DH.$

$\therefore \text{arc } B'C' + \text{arc } DH = 180^\circ.$

But $\text{arc } B'C' = a',$ (Hyp.)

and $\text{arc } DH = \angle A.$ (Proved above)

$\therefore A + a' = 180^\circ,$

or $A = 180^\circ - a'.$

In the same way all the other relations may be proved.

Q.E.D.

717. SCH. Two polar triangles are also called *supplemental triangles*.

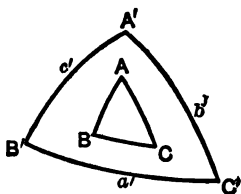
Proposition 14. Theorem.

718. *The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.*

Hyp. Let A, B, C denote the three $\angle$ s of the spherical $\triangle ABC$.

To prove $A + B + C > 180^\circ$ and $< 540^\circ$.

Proof. Let a', b', c' denote the opposite sides respectively of the polar $\triangle A'B'C'$. Then



$$A = 180^\circ - a', B = 180^\circ - b', C = 180^\circ - c'. \quad (716)$$

$$\text{Adding, } A + B + C = 540^\circ - (a' + b' + c').$$

Because a', b', c' are sides of a spherical $\triangle$,

$$\therefore a' + b' + c' < 360^\circ. \quad (705)$$

$$\therefore A + B + C > 180^\circ.$$

Also, because each $\angle$ of the $\triangle$ is < 2 rt. $\angle$ s,

$$\therefore A + B + C < 6 \text{ rt. } \angle\text{s, or } < 540^\circ. \quad \text{Q.E.D.}$$

719. COR. *A spherical triangle may have two, or even three, right angles; also two, or even three, obtuse angles.*

720. If a spherical triangle has two right angles, it is called a *bi-rectangular* triangle; and if a spherical triangle has three right angles, it is called a *tri-rectangular* triangle.

EXERCISE.

In a bi-rectangular triangle the sides opposite the right angles are quadrants.

Proposition 15. Theorem.*

721. *Two mutually equiangular triangles, on the same, or on equal spheres, are mutually equilateral, and are either equal or equivalent.*

Hyp. Let the spherical Δs P and Q be mutually equiangular.

To prove that Δs P and Q are mutually equilateral, and either equal or equivalent.



Proof. Let P' be the polar Δ of P, and Q' the polar Δ of Q.

Since P and Q are mutually equiangular, (Hyp.)

$\therefore$ their polar Δs P' and Q' are mutually equilateral. (716)

And since P' and Q' are mutually equilateral,

$\therefore$ they are mutually equiangular. (706)

But since P' and Q' are mutually equiangular,

$\therefore \Delta s$ P and Q are mutually equilateral. (716)

$\therefore \Delta s$ P and Q are either equal or equivalent. (711)

Q. E. D.

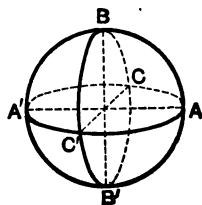
NOTE.—Mutually equiangular spherical triangles are mutually equilateral, only when the triangles are on the same, or on equal spheres. When the spheres are unequal, the homologous sides of the triangles are no longer equal, but are proportional to the radii of their spheres (483); the triangles are then *similar*, as in the case of plane triangles.

722. COR. 1. *If two angles of a spherical triangle are equal, the triangle is isosceles.*

For, since the polar Δ is isosceles (716), the $\angle$ s opp. the two equal sides are equal (712); $\therefore$ the given Δ is isosceles. (716)

723. COR. 2. *If three planes are passed through the centre of a sphere, each perpendicular to the other two, they divide the surface of the sphere into eight equal tri-rectangular triangles.*

(702) and (721)



Proposition 16. Theorem.

724. *In a spherical triangle, the greater side is opposite the greater angle, and conversely.*

(1) *Hyp.* Let ABC be a Δ having $\angle ACB > \angle B$.

To prove $AB > AC$.

Proof. Draw CD , an arc of a great $\odot$, making $\angle BCD = \angle B$.

Then $DB = DC$. (722)

Add AD to each.

Then $AD + DB = AD + DC$.

But $AD + DC > AC$. (704)

$\therefore AD + DB > AC$, or $AB > AC$.

(2) *Hyp.* Suppose $AB > AC$.

To prove $\angle ACB > \angle B$.

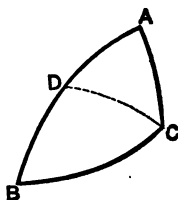
Proof. If $\angle ACB = \angle B$, then $AB = AC$, (722)

which is contrary to the hypothesis.

And if $\angle ACB < \angle B$, then $AB < AC$, (Proved above) which is also contrary to the hypothesis.

$\therefore \angle ACB > \angle B$.

Q. E. D.

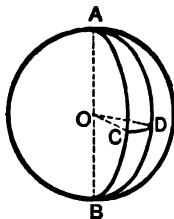


RELATIVE AREAS OF SPHERICAL FIGURES.

DEFINITIONS.

725. A *lune* is a portion of the surface of a sphere included between two semi-circumferences of great circles; as ACBD.

The *angle* of a lune is the angle between the semi-circumferences which form its sides; as the angle CAD, or the angle COD. (699)



726. On the same, or on equal spheres, lunes of equal angles are equal, as they are evidently superposable.

727. A *spherical wedge*, or *ungula*, is the part of a sphere bounded by a lune and the planes of its sides; as AOBCD.

The diameter AB is called the *edge* of the ungula, and the lune ACBD is called its *base*.

728. The *spherical excess* of a spherical triangle is the excess of the sum of its three angles over a straight angle. (718)

Thus, if the angles of a spherical triangle are denoted by A, B, C, and its spherical excess by E, we have

$$E = A + B + C - 1 \text{ st. } \angle.$$

The spherical excess of a spherical polygon is the excess of the sum of its angles over as many straight angles as it has sides, less two.

EXERCISES.

1. Each side of a tri-rectangular spherical triangle is a quadrant. (720)

2. If the three angles of a spherical triangle are 100° , 120° , 95° , find its spherical excess.

3. Given a spherical triangle whose sides are 80° , 100° , 115° : find the angles of its polar triangle.
4. Given a spherical triangle whose angles are 65° , 87° , 115° : find the sides of its polar triangle.
5. Show that the sum of the angles of a spherical pentagon is greater than six, and less than ten, right angles.

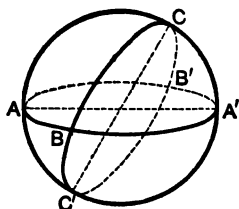
Proposition 17. Theorem.

729. *If two arcs of great circles intersect on the surface of a hemisphere, the sum of the two opposite triangles thus formed is equivalent to a lune whose angle is equal to the angle between the given arcs.*

Hyp. Let the arcs ACA' , BCB' intersect on the surface of the hemisphere $CABA'B'$.

To prove $\triangle ABC + \triangle A'B'C$ equivalent to the lune $CAC'B$.

Proof. Because the $\triangle$ s $A'B'C$ and ABC' are symmetrical, (707)



$$\therefore \text{area } A'B'C = \text{area } ABC' \quad (708)$$

Add area ABC to each.

$$\therefore \text{area } ABC + \text{area } A'B'C = \text{area } ABC + \text{area } ABC' \\ = \text{area of lune } CAC'B.$$

Q.E.D.

730. SCH. It is evident that the two spherical pyramids, which have the triangles ABC , $A'B'C$ for bases, are together equivalent to the spherical wedge whose base is the lune $CAC'B$.

EXERCISE.

If the sides of a triangle are 75° , 110° , and 130° , show that the angles of its polar triangle are respectively 105° , 70° , and 50° .

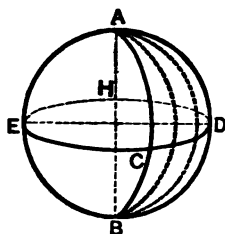
Proposition 18. Theorem.

731. *The area of a lune is to the surface of the sphere as the angle of the lune is to four right angles.*

Hyp. Let ACBD be a lune, and ECDH the great $\odot$ whose poles are A and B; let L denote the area of the lune, S the surface of the sphere, and A the $\angle$ of the lune in degrees.

To prove
$$\frac{L}{S} = \frac{A}{360^\circ}.$$

Proof. Since the $\angle$ of the lune is measured by the arc CD, (698)



$$\therefore A : 360^\circ = \text{arc CD} : \odot \text{ce ECDH}. \quad (236)$$

(1) *When the arc CD and the $\odot$ ce ECDH are commensurable.*

Apply the common measure to the $\odot$ ce ECDH, and let it be contained m times in CD and n times in ECDH.

Then
$$\text{arc CD} : \odot \text{ce ECDH} = m : n.$$

Through the pts. of division of ECDH and the axis AB pass arcs of great $\odot$ s; they will divide the whole surface of the sphere into n equal lunes (726), of which the lune ACBD will contain m .

$$\therefore L : S = m : n.$$

$$\begin{aligned} \therefore L : S &= \text{arc CD} : \odot \text{ce ECDH} \\ &= A : 360^\circ. \end{aligned}$$

(2) *When the arc CD and the $\odot$ ce ECDH are incommensurable.*

The proof in this case may be extended as in (234), (298), and (356). Q.E.D.

732. COR. 1. *Two lunes on the same, or on equal spheres, are to each other as their angles.*

733. COR. 2. If T denote the area of the tri-rectangular Δ (720), 8T will express the surface of the whole sphere. (723)

Then, if the rt. $\angle$ be taken as the unit, we have

$$L : 8T = A : 4. \quad (731)$$

$$\therefore L = 2A \times T.$$

Hence, if the right angle be taken for the unit of angles, the area of a lune is equal to twice its angle multiplied by the area of the tri-rectangular triangle.

734. COR. 3. *The volume of a spherical wedge is to the volume of the sphere as the angle of the lune is to four right angles.*

For, the lunes being equal, the spherical wedges are also equal (727); hence two spherical wedges are to each other as the angles included between their planes.

735. COR. 4. *If the right angle be taken as the unit of angles, the tri-rectangular triangle as the unit of surfaces, and the tri-rectangular pyramid as the unit of volumes, then the area of a lune, and the volume of an ungula, are each expressed by twice its angle.* (733) and (734)

EXERCISES.

1. What part of the surface of a sphere is a lune whose angle is 60° ? 90° ? 120° ?

2. What part of the volume of a sphere is an ungula whose angle is 50° ? 90° ? 100° ?

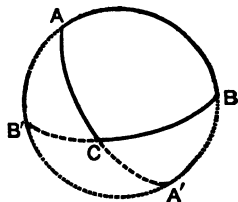
Proposition 19. Theorem.

736. *The area of a spherical triangle is equal to its spherical excess.*

Hyp. Let A, B, C denote the numerical measures of the $\angle$ s of the spherical $\triangle ABC$, the rt. $\angle$ being the unit of $\angle$ s, and the tri-rectangular $\triangle$ the unit of areas.

To prove $\text{area } ABC = A + B + C - 2$.

Proof. Continue any one side as AB so as to complete the great $\odot$ $ABA'B'$. Continue the other two sides AC and BC till they meet this $\odot$ in A' and B' .



Then $\text{area } ABC + \text{area } A'BC = \text{lune } ABA'C = 2A$,

and $\text{area } ABC + \text{area } AB'C = \text{lune } AB'CB = 2B$. (735)

Also, since the $\triangle$ s ABC and $A'B'C$ are together equivalent to a lune whose angle is C , (729)

$\therefore \text{area } ABC + \text{area } A'B'C = 2C$.

But the sum of the areas of the $\triangle$ s $ABC, A'BC, AB'C, A'B'C$ is equal to the area of the hemisphere, or 4.

$\therefore$ adding these three equations, we have

$$2 \text{ area } ABC + 4 = 2A + 2B + 2C.$$

$$\therefore \text{area } ABC = A + B + C - 2. \quad \text{Q.E.D.}$$

737. COR. 1. By a method similar to that in (736), in connection with (730) and (735), it may be proved that

The volume of a triangular spherical pyramid is equal to the spherical excess of its base, (the volume of the tri-rectangular pyramid being the unit of volume).

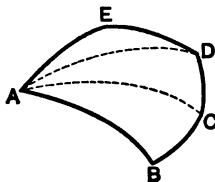
738. COR. 2. If the three vertices of a triangle are on a great circle, its three sides must coincide with that circle, and each angle must equal 180° . The area of the triangle is then a hemisphere, and its spherical excess $4 \text{ rt. } \angle\text{s}$, or 360° . Therefore the area of the surface of the whole sphere is 720° . Hence:

The area of a spherical triangle is to that of the surface of the sphere as its spherical excess, in degrees, is to 720° .

Proposition 20. Theorem.

739. *The area of a spherical polygon is equal to its spherical excess.*

Hyp. Let K denote the area, n the number of sides, and S the sum of the $\angle\text{s}$ of the spherical polygon $ABCDE$, the $\text{rt. } \angle$ and the tri-rectangular Δ being the units of $\angle\text{s}$ and areas respectively.



To prove $K = S - 2(n - 2).$

Proof. From any vertex, as A , draw diagonals, dividing the polygon into $(n - 2) \Delta\text{s}$.

Then, since the area of each $\Delta =$ the sum of its $\angle\text{s}$ minus $2 \text{ rt. } \angle\text{s}$ (736); and since the sum of the $\angle\text{s}$ of the $(n - 2) \Delta\text{s} =$ the sum of the $\angle\text{s}$ of the polygon, or S ,

$$\therefore K = S - 2(n - 2).$$

Q. E. D.

NOTE.—In the last three propositions, only the *ratios* of the areas are expressed. If the *absolute* area is required, the area of the surface of the sphere must be known.

EXERCISES.

THEOREMS.

1. The intersection of the surfaces of two spheres is a circle whose plane is at right angles to the line joining the centres of the spheres, and whose centre is in that line.

2. If lines be drawn from any point of the surface of a sphere to the ends of a diameter, they will form with each other a right angle.

3. If any number of lines in space pass through a point, the feet of the perpendiculars upon these lines from another point lie upon the surface of a sphere.

4. If two straight lines are tangent to a sphere at the same point, the plane of these lines is tangent to the sphere.

5. On spheres of different radii, mutually equiangular triangles are similar.

See (721), note.

6. The sum of the two arcs of great circles drawn from the extremities of one side of a spherical triangle to a point within it, is less than the sum of the other two sides.

7. If from any point on the surface of a sphere two arcs of great circles are drawn perpendicular to a circumference, the shorter of the two arcs is the shortest arc that can be drawn from the given point to the circumference.

8. Any lune is to a tri-rectangular triangle as its angle is to half a right angle.

9. Spherical polygons are to each other as their spherical excesses.

10. Two oblique arcs drawn from the same point to points of the circumference equally distant from the foot of the perpendicular are equal.

11. Of two oblique arcs, the one which meets the circumference at the greater distance from the foot of the perpendicular is the longer.

12. The arc of a great circle, tangent to a small circle, is perpendicular to the radius of the sphere at the point of contact.

13. If three spheres intersect one another, their planes of intersection intersect in a right line perpendicular to the plane containing the centres of the spheres.

14. Prove that this line is the locus of points from which tangent lines to the three spheres are equal.

15. Through a fixed point, within or without a sphere, three lines mutually at right angles intersect the sphere: prove that the sum of the squares of the three chords is constant, depending only on the radius of the sphere and the distance of the point from the centre.

16. Prove also that the sum of the squares of the six segments is constant.

17. Prove that the area of a spherical triangle, each of whose angles is $\frac{1}{4}$ of a right angle, is equal to the surface of a great circle.

18. If through a point O any secant OPP' is drawn to cut a sphere in P, P' , prove that $OP \cdot OP'$ is constant.

19. Find the radii of the spheres inscribed and circumscribed to a regular tetraedron.

NUMERICAL EXERCISES.

20. If the sides of a spherical triangle are respectively 65° , 112° , and 85° , how many degrees are there in each angle of its polar triangle?

21. If the angles of a spherical triangle are respectively 90° , 115° , and 70° , how many degrees are there in each side of its polar triangle?

22. Given the spherical triangle whose sides are respectively 80° , 90° , and 140° , to find the angles of its polar triangle.

23. What part of the surface of a sphere is a lune whose angle is 45° ? 54° ? 80° ?

24. What part of the volume of a sphere is an ungula whose angle is 72° ? 36° ?

25. If the angle of a lune is 50° , find its area on a sphere whose surface is 72 square inches. *Ans.* 10 square inches.

26. Find the area of a spherical triangle whose angles are respectively 75° , 100° , and 115° , on a sphere whose surface is 72 square inches. *Ans.* 11 square inches.

27. Find the area of a spherical triangle each of whose angles is 70° , on a sphere whose surface is 144 square inches. *Ans.* 6 square inches.

28. Find the area of a spherical triangle whose angles are 60° , 90° , and 120° , on a sphere whose surface is 64 square inches. *Ans.* 8 square inches.

29. Find the area of a spherical polygon of six sides each of whose angles is 150° , on a sphere whose surface is 100 square inches. *Ans.* 25 square inches.

30. Find the area of a bi-rectangular triangle whose vertical angle is 108° , on a sphere whose surface is 100 square inches. *Ans.* 15 square inches.

31. Find the area of a spherical pentagon whose angles are respectively 138° , 112° , 131° , 168° , and 153° , on a sphere whose surface is 40 square feet. *Ans.* 9 square feet.

32. Find the area of a spherical triangle whose angles are 61° , 109° , and 127° , on a sphere whose surface is 10 square inches. *Ans.* 1.625 square inches.

33. Find the area of a spherical quadrangle whose angles are 170° , 139° , 126° , and 141° , on a sphere whose surface is 400 square inches.

34. Find the area of a spherical pentagon whose angles are 122° , 128° , 131° , 160° , and 161° , on a sphere whose surface is 150 square feet.

35. Find the angles of an equilateral spherical triangle whose area is equal to that of a great circle. *Ans.* 120° .

36. Find the angles of an equilateral spherical triangle whose area is equal to that of an equilateral spherical hexagon, each of whose angles is 150° . *Ans.* 80° .

37. Find the volume of a triangular spherical pyramid, the angles of whose bases are 80° , 90° , 130° ; the volume of the sphere being 30 cubic inches.

Ans. 5 cubic inches.

38. Find the volume of a quadrangular spherical pyramid, the angles of whose bases are 170° , 139° , 126° , 141° ; the volume of the sphere being 10 cubic inches.

Ans. 3 cubic inches.

39. Find the ratio of the areas of two spherical triangles on the same sphere, the angles being 60° , 84° , 129° , and 83° , 107° , 114° respectively.

40. Find the circumference of a small circle of a sphere whose diameter is 10 inches, the plane of the circle being 3 inches from the centre of the sphere.

Ans. 25.133 inches.

41. Find the circumference of a small circle of a sphere whose diameter is 20 inches, the plane of the circle being 4 inches from the centre of the sphere.

42. Find the area of the circle of intersection of two spheres, their radii being 4 and 6 inches and the distance between their centres 5 inches.

43. Find the area of a small circle of a sphere whose diameter is 5 inches, the plane of the circle being 1 inch from the centre of the sphere.

44. The radii of two concentric spheres are 4 and 6 inches, a plane is drawn tangent to the interior sphere: find the area of the section made in the outer sphere.

45. Given two mutually equiangular triangles on spheres whose radii are 10 and 16 inches: find the ratio of two homologous sides of these triangles.

See (721), *note*.

PROBLEMS.

46. To pass a plane tangent to a sphere at a given point on the surface of the sphere.

47. To pass a plane tangent to a sphere through a given straight line without the sphere.

48. To construct on the spherical blackboard a spherical angle of 45° ; 60° ; 90° ; 100° ; 200° .

From P, the pt. where the vertex is to be placed, with a quadrant describe an arc, which will represent one side of the angle required. From P as a pole, with a quadrant describe an arc from the side before drawn, to measure the required angle. Lay off on this last arc from the first arc the measure of the required angle; and through the extremity of this arc and P pass a great circle. For describing the arcs, the student can use a tape equal in length to half a great circle of the sphere, marked off into 180 equal parts.

49. To construct on the spherical blackboard a spherical triangle, having two sides 100° and 80° , and the included angle 58° .

50. To construct, as above, a spherical triangle, having a side 75° , and the adjacent angles 110° and 87° .

51. To construct, as above, a spherical triangle, having its sides 150° , 100° , 80° ; also having its sides 50° , 85° , 160° .

52. To construct, as above, a spherical triangle, having two sides 120° and 88° , and the included angle 59° ; then construct its polar triangle.

53. Given two points on the surface of a sphere, to describe the great circle passing through them.

54. To bisect a given arc, or a given angle, on a sphere.

55. To draw an arc of a great circle perpendicular to a spherical arc, from a given point without it.

56. To erect a perpendicular to a given arc of a great circle from a given point in the arc.

57. Given three points on a sphere, to describe a small circle to pass through them.

58. To cut a given sphere by a plane passing through a given straight line so that the section shall have a given radius.

59. At a given point in a great circle, to draw an arc of a great circle making a given angle with the first.

60. Through a given point on a sphere, to draw a great circle tangent to a given small circle.

61. Through a given point on a sphere, to draw a great circle tangent to two given small circles.

62. To inscribe a circle in a given spherical triangle, and to circumscribe a circle about the triangle.

63. To construct a right spherical triangle, having (1) a side about the right angle and the hypotenuse; and (2) an angle and the opposite side.

64. To construct a spherical triangle, having given (1) the three sides; and (2) two sides and the included angle.

65. To describe a sphere to cut orthogonally two given spheres.

BOOK IX.*

THE THREE ROUND BODIES.

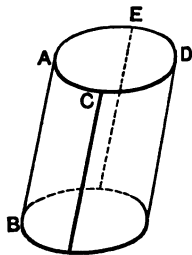
740. The only solids bounded by curved surfaces, that are treated of in Elementary Geometry, are the *cylinder*, the *cone*, and the *sphere*, which are called the *three round bodies*.

THE CYLINDER.

DEFINITIONS.

741. A *cylindrical surface* is a surface generated by the motion of a straight line AB, called the *generatrix*, which constantly touches a given curve ACDE, called the *directrix*, and remains parallel to its original position. The different positions of the generatrix are called *elements* of the surface.

742. A *cylinder* is a solid bounded by a cylindrical surface and two parallel planes. The cylindrical surface is called the *lateral surface*, and the plane surfaces are called the *bases*.



The *altitude* of a cylinder is the perpendicular distance between its bases.

743. A *right section* of a cylinder is the section by a plane perpendicular to its elements.

744. A *circular cylinder* is a cylinder whose base is a circle.

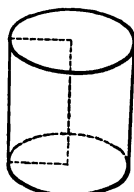
The *axis* of a circular cylinder is the straight line joining the centres of its bases.

745. A *right cylinder* is one whose elements are perpendicular to its bases.

* This book treats of the properties and relations of the *cylinder*, the *cone*, and the *sphere*, and shows how to find the *convex surface* and *volume* of each of these bodies.

746. A *right circular cylinder*, called also a *cylinder of revolution*, is generated by revolving a rectangle about one of its sides.

747. Similar cylinders of revolution are those generated by similar rectangles revolving round homologous sides.



748. A *tangent plane* to a cylinder is a plane which contains an element of the cylinder without cutting the surface. The element which the plane contains is called the *element of contact*.

Any straight line in a tangent plane, which cuts the element of contact, is a *tangent line* to the cylinder.

749. A prism is *inscribed* in a cylinder, when its bases are inscribed in the bases of the cylinder and its lateral edges are elements of the cylinder.

750. A prism is *circumscribed* about a cylinder, when its bases are circumscribed about the bases of the cylinder.

Proposition 1. Theorem.

751. Every section of a cylinder made by a plane passing through an element is a parallelogram.

Hyp. Let the plane ABCD pass through the element AB of cylinder EH.

To prove the section ABCD a $\square$.

Proof. A plane passing through the element AB cuts the $\odot$ ce of the base in a second pt. D.

Through D draw DC $\parallel$ to AB.

Then DC is in the plane BAD. (68)

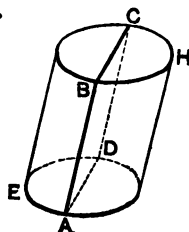
$\therefore$ DC is an element of the cylinder. (741)

$\therefore$ DC, being common to the plane and the lateral surface of the cylinder, is their intersection.

Also, AD is $\parallel$ to BC. (519) $\therefore$ ABCD is a $\square$. (124)

Q. E. D.

752. COR. Every section of a right cylinder made by a plane passing through an element is a rectangle.



Proposition 2. Theorem.

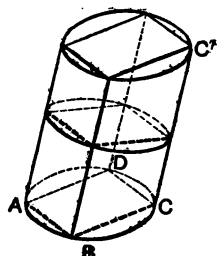
753. *The lateral area of a cylinder is equal to the perimeter of a right section of the cylinder multiplied by an element.*

Hyp. Let S denote the lateral area, P the perimeter of a rt. section, and E an element of the cylinder AC' .

To prove $S = P \times E$.

Proof. Inscribe in the cylinder a prism $ABCD-C'$, of any number of faces; and let s denote its lateral area and p the perimeter of its rt. section.

Then, since each lateral edge is an element of the cylinder,



(749)

$$\therefore s = p \times E. \quad (591)$$

Now let the number of lateral faces of the inscribed prism be indefinitely increased.

The perimeter of the right section of the prism will approach the perimeter of the right section of the cylinder as its limit. (430)

$\therefore$ the lateral area of the prism will approach the lateral area of the cylinder as its limit.

Because, however great the number of the lateral faces,

$$s = p \times E,$$

and because p approaches P as its limit and s approaches S as its limit,

$$\therefore S = P \times E. \quad \text{Q. E. D.}$$

754. COR. 1. *The lateral area of a cylinder of revolution is equal to the circumference of its base multiplied by its altitude.*

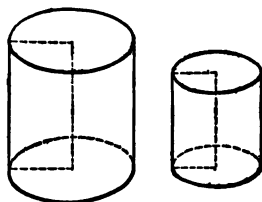
755. COR. 2. If H denote the altitude of a cylinder of revolution, R the radius of the base, S the lateral area, and T the total area, we have

$$S = 2\pi R \times H, \quad (436)$$

$$T = 2\pi RH + 2\pi R^2 = 2\pi R(R + H). \quad (439)$$

756. COR. 3. Let S, S' denote the lateral areas; T, T' the total areas; R, R' the radii of the bases; and H, H' the altitudes of two similar cylinders of revolution.

Then, since the generating rectangles are similar, (747)



$$\therefore \frac{R}{R'} = \frac{H}{H'} = \frac{R + H}{R' + H'}.$$

$$\therefore \frac{S}{S'} = \frac{2\pi RH}{2\pi R'H'} = \frac{R}{R'} \times \frac{H}{H'} = \frac{H^2}{H'^2} = \frac{R^2}{R'^2},$$

$$\text{and } \frac{T}{T'} = \frac{R(R + H)}{R'(R' + H')} = \frac{R}{R'} \cdot \frac{R + H}{R' + H'} = \frac{H^2}{H'^2} = \frac{R^2}{R'^2}. \quad (755)$$

Therefore, the lateral areas, or the total areas, of two similar cylinders of revolution are to each other as the squares of their altitudes, or as the squares of the radii of their bases.

EXERCISES.

1. Required the lateral area, and also the total area, of a cylinder of revolution whose altitude is 25 inches and the diameter of whose base is 20 inches.

2. Required the convex surface of a right circular cylinder whose altitude is 16 inches, and diameter of the base 8 inches.

3. Required the altitude and radius of the base of a right circular cylinder whose lateral area is $\frac{3}{4}$ as great as a similar cylinder of which the altitude is 20 feet and diameter of the base 8 feet.

Proposition 3. Theorem.

757. *The volume of a cylinder is equal to the product of its base by its altitude.*

Hyp. Let V , B , H denote the volume of the cylinder, the area of its base, and its altitude, respectively.

To prove $V = B \times H$.

Proof. Inscribe in the cylinder a prism, and let V' and B' denote its volume and the area of its base.

Then, since the altitude of the prism is H ,

$$\therefore V' = B' \times H. \quad (612)$$

Now let the number of lateral faces of the inscribed prism be indefinitely increased.

The base of the prism B' will approach the base of the cylinder B as its limit, and the volume of the prism V' will approach the volume of the cylinder V as its limit.

Because, however great the number of the lateral faces, we always have

$$V' = B' \times H.$$

$$\therefore V = B \times H. \quad \text{Q.E.D.}$$

758. COR. 1. *If R denotes the radius of the base of a cylinder of revolution, then $B = \pi R^2$.* (439)

$$\therefore V = \pi R^2 \cdot H. \quad (757)$$

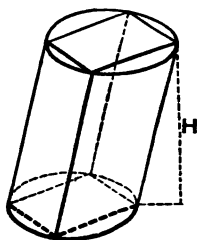
759. COR. 2. Let V , V' denote the volumes, R , R' the radii of the bases, and H , H' the altitudes of two similar cylinders of revolution.

Then, since the generating rectangles are similar, (747)

$$\therefore \frac{R}{R'} = \frac{H}{H'}.$$

$$\therefore \frac{V}{V'} = \frac{R^2 H}{R'^2 H'} = \frac{R^2}{R'^2} \times \frac{H}{H'} = \frac{H^2}{H'^2} = \frac{R^2}{R'^2}. \quad (758)$$

Therefore, the volumes of similar cylinders of revolution are to each other as the cubes of their altitudes, or as the cubes of their radii.



THE CONE.

DEFINITIONS.

760. A *conical surface* is a surface generated by the motion of a straight line SA , called the *generatrix*, which constantly touches a given curve $ABCD$, called the *directrix*, and always passes through a fixed point S , called the *vertex*. The different positions of the generatrix are called *elements* of the surface.

If the generatrix extends on both sides of the vertex, the whole surface consists of two portions, $S-ABCD$ and $S-abcd$, lying on opposite sides of the vertex, which are called the *lower* and *upper nappes*, respectively.

761. A *cone* is a solid bounded by a conical surface and a plane cutting all its elements. The conical surface is called the *lateral surface*.

The *base* of a cone is its plane surface.

The *altitude* of a cone is the perpendicular distance from the vertex to the base.

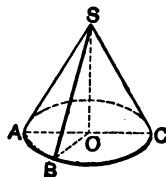
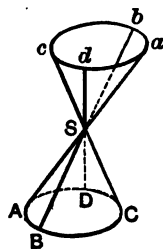
762. A *circular cone* is a cone whose base is a circle.

The *axis* of a circular cone is the straight line joining the vertex and the centre of its base.

763. A *right cone* is one whose axis is perpendicular to its base.

The cones principally treated of in Elementary Geometry are right cones.

764. A *right circular cone* is a circular cone whose axis is perpendicular to its base. It is also called a *cone of revolution*, because it may be generated by the revolution of a right triangle SOA , about one of its perpendicular sides SO , as an axis: the hypotenuse SA generates the lateral surface, and OA generates the base. All the elements SA ,



SB, etc., are equal; and any one is called the *slant height* of the cone.

765. *Similar cones of revolution* are those generated by similar right triangles revolving round homologous perpendicular sides.

766. A *tangent plane* to a cone is a plane which contains an element of the cone without cutting the surface. The element which the plane contains is called the *element of contact*.

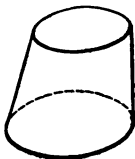
Any straight line in a tangent plane, which cuts the element of contact, is a *tangent line* to the cone.

767. A *frustum* of a cone is the portion of a cone included between its base and a plane parallel to the base and cutting all the elements.

If the cutting plane is not parallel to the base, the portion included between the base and plane is called a *truncated cone*.

768. The *altitude* of a frustum of a cone is the perpendicular distance between its bases.

The *slant height* of the frustum of a cone of revolution is the portion of an element included between the parallel bases of the frustum.



769. A pyramid is *inscribed* in a cone when its base is inscribed in the base of the cone, and its lateral edges are elements of the cone.

Any plane which cuts the cone, determines a truncated pyramid inscribed in the truncated cone.

770. A pyramid is *circumscribed* about a cone when its base is circumscribed about the base of the cone, and its vertex coincides with the vertex of the cone.

Any plane which cuts the cone, determines a truncated pyramid circumscribed about the truncated cone.

Proposition 4. Theorem.

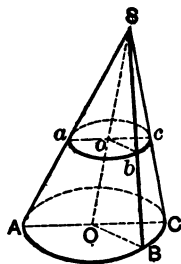
771. Every section of a circular cone made by a plane parallel to the base is a circle.

Hyp. Let the section abc of the circular cone S - ABC be $\parallel$ to the base.

To prove that abc is a $\odot$.

Proof. Let SO be the axis of the cone, cutting the plane of the $\parallel$ section in the pt. o .

Through SO and any elements SA , SB , pass planes cutting the base in the radii OA , OB , and the $\parallel$ section in the st. lines oa , ob .



Then, since the planes abc and ABC are $\parallel$, (Hyp.)

$\therefore oa$ is $\parallel$ to OA , and ob is $\parallel$ to OB . (519)

$$\therefore \frac{oa}{OA} = \frac{So}{SO}, \text{ and } \frac{ob}{OB} = \frac{So}{SO}.$$

$$\therefore \frac{oa}{OA} = \frac{ob}{OB}.$$

But $OA = OB$. (Radii)

$$\therefore oa = ob.$$

Hence, all the st. lines drawn from the pt. o to the perimeter of the section abc are equal.

$\therefore$ the section is a $\odot$. Q.E.D.

772. The axis of a circular cone passes through the centres of all the sections parallel to the base.

EXERCISES.

1. Prove that every section of a cone made by a plane passing through its vertex is a triangle.

2. Required the volume of a circular cylinder whose altitude is 25 inches and diameter of the base 20 inches.

Proposition 5. Theorem.

773. *The lateral area of a cone of revolution is equal to the product of the circumference of its base by half its slant height.*

Hyp. Let S , C , L denote the lateral area of the cone, the $\odot$ ce of its base, and its slant height, respectively.

To prove $S = C \times \frac{1}{2}L$.

Proof. Inscribe in the cone a pyramid $S-ABCD$, having a regular base of any number of sides; and let S' denote its lateral area, C' the perimeter of its base, and L' its slant height.

Then, since the edges of the pyramid are elements of the cone (769), the pyramid is regular.

$$\therefore S' = C' \times \frac{1}{2}L'. \quad (628)$$

Now let the number of lateral faces of the inscribed pyramid be indefinitely increased.

C' will approach C as its limit. (430)

$\therefore L'$ and S' will approach L and S respectively, as their limits.

Because, however great the number of lateral faces,

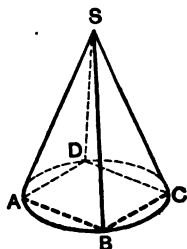
$$S' = C' \times \frac{1}{2}L',$$

$$\therefore S = C \times \frac{1}{2}L. \quad \text{Q.E.D.}$$

774. COR. 1. *If R is the radius of the base of a cone of revolution, and T is the total area, we have*

$$S = 2\pi R \times \frac{1}{2}L = \pi RL. \quad (436)$$

$$T = \pi RL + \pi R^2 = \pi R (L + R).$$



775. COR. 2. By the process that was employed in (756) we may show that *the lateral areas, or the total areas, of two similar cones of revolution are to each other as the squares of their radii, or of their slant heights, or of their altitudes.*

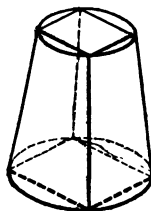
Proposition 6. Theorem.

776. *The lateral area of a frustum of a cone of revolution is equal to half the sum of the circumferences of its bases multiplied by its slant height.*

Hyp. Let S , C , c , L denote the lateral area of the frustum, the $\odot$ ces of its bases, and its slant height, respectively.

To prove $S = \frac{1}{2}(C + c)L$.

Proof. Inscribe in the frustum of the cone the frustum of a regular pyramid; and let S' denote its lateral area, C' and c' the perimeters of its lower and upper bases, respectively, and L' its slant height.



$$\text{Then} \quad S' = \frac{1}{2}(C' + c')L'. \quad (629)$$

Now let the number of lateral faces of the inscribed frustum be indefinitely increased.

C' and c' will approach C and c respectively, as their limits. (430)

$\therefore L'$ and S' will approach L and S respectively, as their limits.

$$\therefore S = \frac{1}{2}(C + c)L. \quad \text{Q.E.D.}$$

777. COR. *The lateral area of a frustum of a cone of revolution is equal to the circumference of a section equidistant from its bases* multiplied by its slant height.*

* Called the mid-section.

Proposition 7. Theorem.

778. *The volume of any cone is equal to one-third the product of its base and altitude.*

Hyp. Let V , B , H denote the volume of the cone, the area of its base, and its altitude, respectively.

To prove $V = \frac{1}{3}B \times H$.

Proof. Inscribe in the cone a pyramid, and let V' and B' denote its volume and the area of its base.

Then $V' = \frac{1}{3}B' \times H$. (632)

Now let the number of lateral faces of the pyramid be indefinitely increased.

B' will approach B as its limit. (430)

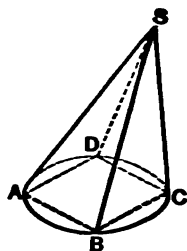
$\therefore V'$ will approach V as its limit.

$\therefore V = \frac{1}{3}B \times H$. Q.E.D.

779. COR. 1. *For the volume of a cone of revolution, whose altitude is H and radius of the base is R , we have*

$$V = \frac{1}{3}\pi R^2 H. \quad (439)$$

780. COR. 2. *The volumes of similar cones of revolution are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.* (759)



EXERCISES.

1. Required the lateral area and volume of a right circular cone whose altitude is 24 inches and radius of the base 10 inches. *Ans.* 816.82 sq. ins.; 2513.3 cu. ins.

2. Required the entire surface of a right circular cone whose altitude is 16 inches and radius of the base 12 inches.

Ans. 1206.37 sq. ins.

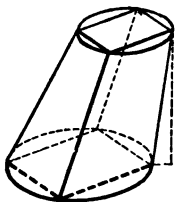
Proposition 8. Theorem.

781. *The volume of a frustum of any cone is equal to the sum of the volumes of three cones, whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum.*

Hyp. Let V , B , b , H denote the volume of the frustum, its bases, and its altitude, respectively.

To prove $V = \frac{H}{3}(B + b + \sqrt{Bb}).$

Proof. Inscribe in the frustum of the cone the frustum of a pyramid, and let V' , B' , b' denote its volume and the areas of its bases.



Then $V' = \frac{H}{3}(B' + b' + \sqrt{B'b'}). \quad (635)$

Now let the number of lateral faces of the inscribed frustum be indefinitely increased.

B' and b' will approach B and b respectively, as their limits. (430)

$\therefore V'$ will approach V as its limit.

$$\therefore V = \frac{H}{3}(B + b + \sqrt{Bb}). \quad \text{Q. E. D.}$$

782. COR. 1. *If the frustum is that of a right circular cone, and the radii of its bases are R and r , we have $B = \pi R^2$, $b = \pi r^2$.*

$$\therefore V = \frac{1}{3}\pi H(R^2 + r^2 + Rr).$$

783. COR. 2. *This formula may be put into the form*

$$V = \pi H \left[\left(\frac{R+r}{2} \right)^2 + \frac{1}{3} \left(\frac{R-r}{2} \right)^2 \right].$$

NOTE.—The volume of a cask may be found approximately by this formula, in which H = total height of cask, R = radius of mid-section, and r = radius of end.

The nearest approximation in the case of most casks is given by the formula

$$V = \frac{1}{2}\pi H[2R^2 + r^2 - \frac{1}{4}(R^2 - r^2)].^*$$

* See Rouché et Comberousse, p. 155.

THE SPHERE.

DEFINITIONS.

784. A *zone* is a portion of the surface of a sphere included between two parallel planes.

The *altitude* of the zone is the perpendicular distance between the parallel planes.

The *bases* of the zone are the circumferences of the circles which bound the zone.

If one of the parallel planes touches the sphere, the zone is called a *zone of one base*.

785. A *spherical segment* is a portion of the *volume* of a sphere included between two parallel planes.

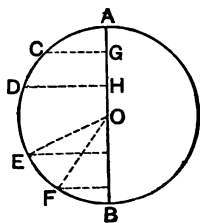
The *altitude* of the segment is the perpendicular distance between the parallel planes.

The *bases* of the segment are the sections of the sphere made by the parallel planes.

A *segment of one base* is a segment one of whose bounding planes touches the sphere.

786. A *spherical sector* is a portion of the volume of a sphere generated by the revolution of a circular sector about a diameter of the circle.

787. Let the sphere be generated by the revolution of the semicircle ACDEFB about its diameter AB as an axis; and let CG and DH be drawn perpendicular to the axis. The arc CD generates a zone whose altitude is GH, and the figure CDHG generates a spherical segment whose altitude is GH. The circumferences generated by the points C and D are the bases of the zone, and the circles generated by CG and DH are the bases of the segment.



The arc AC generates a zone of one base, and the figure ACG a spherical segment of one base.

The circular sector OEF generates a spherical sector whose base is the zone generated by the arc EF; the other bounding surfaces are the conical surfaces generated by the radii OE and OF.

If OF coincides with OB, the spherical sector is bounded by a conical surface and a zone of one base.

If OE is perpendicular to OB, the spherical sector is bounded by a plane surface, a conical surface, and a zone.

Proposition 9. Theorem.

788. *The area generated by a straight line revolving about an axis in its plane, is equal to the product of the projection of the line on the axis by the circumference whose radius is the perpendicular erected at the middle point of the line and terminated by the axis.*

Hyp. Let AB be the revolving line, CD its projection on the axis GO, and EO the $\perp$ at the mid. pt. of AB and terminating in the axis.

To prove area AB = CD $\times$ 2π EO.

Proof. Draw EF $\perp$, and AH $\parallel$ to GO.

The area generated by AB is the lateral area of a frustum of a cone of revolution, whose slant height is AB and axis CD.

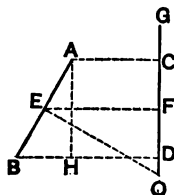
$$\therefore \text{area AB} = AB \times 2\pi EF. \quad (777)$$

The $\triangle$ s ABH and EOF are similar. (316)

$$\therefore AB \times EF = AH \times EO = CD \times EO.$$

$$\therefore \text{area AB} = CD \times 2\pi EO.$$

If AB meets the axis, or is $\parallel$ to it, thus generating a conical, or a cylindrical surface, the result is the same from (773) and (753). Q.E.D.



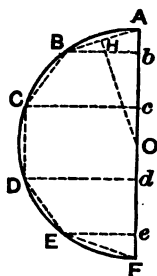
Proposition 10. Theorem.

789. *The area of the surface of a sphere is equal to the product of its diameter by the circumference of a great circle.*

Hyp. Let the sphere be generated by the revolution of the semicircle ABDF about the diameter AF, let O be the centre, R the radius, and denote the surface of the sphere by S.

To prove $S = AF \times 2\pi R.$

Proof. Inscribe in the semicircle a regular semi-polygon ABCDEF, of any number of sides.



Draw Bb, Cc , etc., $\perp$ to AF and $OH \perp$ to AB.

OH bisects AB.

(201)

Then $\text{area } AB = Ab \times 2\pi OH.$

(788)

Similarly, $\text{area } BC = bc \times 2\pi OH$, and so on.

In equal & equal chords are equally distant from the centre (206).

Now the sum of the projections Ab, bc , etc., of all the sides of the semi-polygon make up the diameter AF.

$\therefore$ the entire surface generated by the revolving semi-polygon

$$= AF \times 2\pi OH.$$

Now let the number of sides of the inscribed semi-polygon be indefinitely increased.

The semi-perimeter will approach the semi-circumference as its limit, and OH will approach the radius R as its limit.

(430)

$\therefore$ the surface of revolution will approach the surface of the sphere as its limit.

$$\therefore S = AF \times 2\pi R.$$

Q.E.D.

790. COR. 1. Since $AF = 2R$,

$$\therefore S = 2R \times 2\pi R = 4\pi R^2. \quad (789)$$

Therefore, *the area of the surface of a sphere is equal to the area of four great circles.*

791. COR. 2. *The areas of the surfaces of two spheres are to each other as the squares of their radii, or as the squares of their diameters.*

792. COR. 3. *The area of a zone is equal to the product of its altitude by the circumference of a great circle.*

For, the area of the zone generated by the revolution of the arc BD

$$= bd \times 2\pi R.$$

793. COR. 4. *Zones on the same sphere, or on equal spheres, are to each other as their altitudes.*

794. COR. 5. Since the arc AB generates a zone of one base whose area is

$$Ab \times 2\pi R = \pi Ab \times AF = \pi \overline{AB}^2, \quad (325)$$

therefore, *the area of a zone of one base is equal to the area of the circle whose radius is the chord of the zone.*

795. COR. 6. If a cylinder is circumscribed about a sphere, the total area of the cylinder $= 6\pi R^2$. (755)

Therefore, *the area of the surface of a sphere is equal to two-thirds the total area of the circumscribing cylinder.*

EXERCISES.

1. Find the area of the surface of a sphere whose radius is 4 inches.

2. Prove that two zones on different spheres are to each other as the products of their altitudes by the radii of the spheres.

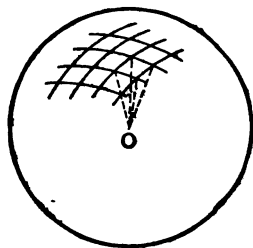
Proposition 11. Theorem.

796. *The volume of a sphere is equal to the area of its surface multiplied by one-third of its radius.*

Hyp. Let V denote the volume of a sphere, S the area of its surface, and R its radius.

To prove $V = S \times \frac{1}{3}R$.

Proof. Conceive the whole surface of the sphere to be divided into a great number of equal spherical polygons. Form pyramids by joining the vertices of these polygons together successively and to the centre of the sphere. It is evident that these pyramids will have a common altitude.



The volume of each pyramid = base $\times \frac{1}{3}$ altitude. (631)

$\therefore$ the sum of all the pyramids = sum of bases $\times \frac{1}{3}$ altitude.

Now let the number of spherical polygons be indefinitely increased.

The sum of the bases of all the pyramids will approach the surface of the sphere as its limit; and the altitude of each pyramid will approach the radius of the sphere as its limit.

$\therefore$ the sum of the volumes of all the pyramids will approach the volume of the sphere as its limit.

$$\therefore V = S \times \frac{1}{3}R.$$

Q. E. D.

NOTE.—This result might be obtained by regarding the sphere as the limit of a circumscribed polyedron, the number of whose faces was indefinitely increased. For, if pyramids are formed having the faces of the polyedron as their bases, and the centre of the sphere as their common vertex, these pyramids will have a common altitude equal to the radius of the sphere.

Then each pyramid = face $\times \frac{1}{3}$ altitude.

$\therefore$ sum of pyramids = sum of faces $\times \frac{1}{3}$ altitude.

But in the limit sum of faces of polyedron = surface of sphere: and sum of volumes of pyramids = volume of sphere.

$$\therefore V = S \times \frac{1}{3}R.$$

797. COR. 1. *The volume of a spherical pyramid is equal to the area of its base multiplied by one-third the radius of the sphere.* (796)

798. COR. 2. Since $S = 4\pi R^2$, (790)

$$\therefore V = \frac{1}{3}\pi R^3; \quad (796)$$

or, if D denotes the diameter,

$$V = \frac{1}{6}\pi D^3.$$

799. COR. 3. *The volumes of two spheres are to each other as the cubes of their radii.*

800. COR. 4. *The volume of a spherical sector is equal to the area of the zone which forms its base multiplied by one-third the radius of the sphere.*

For, a spherical sector, like the entire sphere, may be conceived as consisting of an indefinitely great number of pyramids whose bases make up its surface, and whose common altitude is the radius of the sphere.

801. COR. 5. The volume of the cylinder circumscribed about a sphere $= 2\pi R^3$.

Therefore, *the volume of a sphere is equal to two-thirds the volume of the circumscribing cylinder.*

NOTE.—It was Archimedes who discovered that the surface and volume of the sphere are each two-thirds of that of the circumscribing cylinder.

EXERCISES.

1. Show that the volumes of spherical sectors of the same sphere, or equal spheres, are to each other as the altitudes of the zones which form their bases.

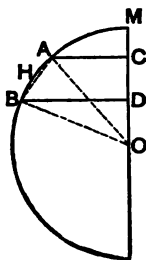
2. Find the volume of a spherical sector, if the altitude of the zone which forms its base is 3 feet, and the radius of the sphere is 6 feet.

Proposition 12. Theorem.

802. *The volume of a spherical segment is equal to half the sum of its bases multiplied by its altitude, plus the volume of a sphere of which this altitude is the diameter.*

Hyp. Let AB be the arc of a $\odot$, and CD the projection of the chord AB on the diameter OM.

Let $AC = r'$, $BD = r$, $CD = h$, $OA = R$, and denote the volume of the segment generated by revolving the circular segment AHBDC about OM by V.



To prove $V = \frac{1}{2}\pi h(r^2 + r'^2) + \frac{1}{6}\pi h^3$.

Proof. Draw the radii OA, OB.

The volume generated by AHBDC is the sum of the spherical sector generated by OAHB and the cone generated by OAC, diminished by the cone generated by OBD.

$$\text{Vol. sph. sect. OAHB} = \frac{2}{3}\pi R^2 h. \quad (800)$$

$$\text{Vol. of cone OAC} = \frac{1}{3}\pi r'^2 OC.$$

$$\text{Vol. of cone OBD} = \frac{1}{3}\pi r^2 OD. \quad (778)$$

$$\therefore V = \frac{2}{3}\pi R^2 h + \frac{1}{3}\pi r'^2 OC - \frac{1}{3}\pi r^2 OD.$$

$$\text{But } h = OC - OD, R^2 - r'^2 = \overline{OC}^2, \text{ and } R^2 - r^2 = \overline{OD}^2.$$

$$\begin{aligned} \therefore V &= \pi R^2 h - \frac{1}{3}\pi(\overline{OC}^2 - \overline{OD}^2) \\ &= \frac{1}{3}\pi h[3R^2 - (\overline{OC}^2 + OC \cdot OD + \overline{OD}^2)]. \end{aligned}$$

$$\text{Since } OC - OD = h,$$

$$\therefore \overline{OC}^2 - 2OC \cdot OD + \overline{OD}^2 = h^2, \quad \text{and}$$

$$\begin{aligned} \therefore \overline{OC}^2 + OC \cdot OD + \overline{OD}^2 &= \frac{2}{3}(\overline{OC}^2 + \overline{OD}^2) - \frac{h^2}{2} \\ &= 3R^2 - \frac{2}{3}(r^2 + r'^2) - \frac{h^2}{2}. \end{aligned}$$

$$\begin{aligned} \therefore V &= \frac{1}{3}\pi h \left[\frac{2}{3}(r^2 + r'^2) + \frac{h^2}{2} \right] \\ &= \frac{1}{2}\pi h(r^2 + r'^2) + \frac{1}{6}\pi h^3. \end{aligned}$$

Q. E. D.

803. COR. 1. If the segment has but one base, as the volume generated by MABD, the radius $r' = 0$, and we have

$$V = \frac{1}{3}\pi r^2 h + \frac{1}{3}\pi h^3.$$

Therefore, *the volume of a spherical segment of one base is equal to half the cylinder having the same base and the same altitude, plus the sphere of which this altitude is the diameter.*

804. COR. 2. When the segment has but one base,

$$r^2 = (2R - h)h.$$

$$\therefore V = \frac{1}{3}\pi h^2(2R - h) + \frac{1}{3}\pi h^3 \quad (803)$$

$$= \pi h^2(R - \frac{1}{3}h).$$

805. COR. 3. Let V' denote the volume of a frustum of a cone generated by the trapezoid ABDC about OM, and v the volume generated by the circular segment AHB.

$$\text{Then} \quad V' = \frac{1}{3}\pi h(r^2 + r'^2 + rr'), \quad (782)$$

$$\text{and} \quad V = \frac{1}{3}\pi h(r^2 + r'^2) + \frac{1}{3}\pi h^3. \quad (802)$$

$$\text{Subtracting, } v = \frac{1}{3}\pi h(r^2 + r'^2 + h^2 - 2rr')$$

$$= \frac{1}{3}\pi \overline{AB}^2 \cdot h.$$

Therefore, *the volume generated by a circular segment revolving about a diameter exterior to its surface, is equal to one-sixth of the cylinder whose radius is the chord of the segment and whose altitude is the projection of this chord on the axis.*

EXERCISES.

THEOREMS.

1. The lateral area of a cylinder of revolution is equal to the area of a circle whose radius is a mean proportional between the altitude of the cylinder and the diameter of its base.

2. If the slant height of a cone of revolution is equal to the diameter of its base, its lateral area is double the area of its base.

3. The volume of a cylinder of revolution is equal to the product of its lateral area by half its radius.

4. A plane through two elements of a cylinder of revolution cuts the base in a chord which subtends at its centre an angle of $\frac{\pi}{3}$: compare the lateral areas of the two parts of the cylinder.

5. A rectangle revolves successively about two adjacent sides whose lengths are a and b : compare the volumes of the two cylinders that are generated.

6. The two legs of a right triangle are a and b : find the area of the surface generated when the triangle revolves about its hypotenuse.

7. Prove that a sphere may be inscribed in a cylinder of revolution, and that it will touch it along the circumference of a great circle.

8. The lateral area of a given cone of revolution is double the area of its base: find the ratio of its altitude to the radius of its base.

9. On each base of a frustum of a cone of revolution, a cone stands having its vertex in the centre of the other base: find the radius of the circle of intersection of the two cones, the radii of the bases being r_1 and r_2 .

10. If the altitude of a cylinder of revolution be equal to the diameter of its base,* the volume is equal to the product of its total area by one-third of its radius.

11. If the slant height of a cone of revolution be equal to the diameter of its base,† its total area is to the area of the inscribed sphere as 9 : 4.

12. In a frustum of a cone of revolution the inclination of the slant height to one base is 45° : find the lateral area, the radii of the bases being r_1 and r_2 .

13. If the radius of a sphere is bisected at right angles by a plane, the two zones into which the surface of the sphere is divided are to each other as 3 : 1.

14. If a cylinder and cone, each equilateral, be inscribed in a sphere, the total area of the cylinder is a mean proportional between the total area of the cone and the area of the sphere. The same is true of the volumes of these bodies.

15. If a cylinder and cone, each equilateral, be circumscribed about a sphere, the total area of the cylinder is a mean proportional between the total area of the cone and the area of the sphere. The same is true of the volumes.

16. A cone of revolution whose vertical angle is 60° , is circumscribed about a sphere: compare the area of the sphere and the lateral area of the cone. Compare their volumes.

17. The base of a cone is equal to a great circle of a sphere, and the altitude of the cone is equal to a diameter of the sphere: compare the volumes of the cone and sphere.

18. The volume of a cone of revolution is equal to the area of its generating rectangle multiplied by the circumference generated by the point of intersection of the diagonals of the rectangle.

19. The volume of a sphere is to the volume of the circumscribed cube as $\pi : 6$.

* Called an *equilateral cylinder*.

† An *equilateral cone*.

20. The volume of a sphere is to the volume of the inscribed cube as $\pi : 2$.

21. If d is the distance of a point P from the centre of a sphere whose radius is R , the sum of the squares of the six segments of three chords at right angles to each other passing through P is $6R^2 - 2d^2$.

22. If h is the height of an aeronaut, and R is the radius of the earth, the extent of surface visible = $\frac{2\pi R^2 h}{R + h}$.

NUMERICAL EXERCISES.

23. Find the lateral area, total area, and volume, of a cylinder of revolution, the radius of the base being 4 and the altitude 10.

24. Find the lateral area, total area, and volume, of a cone of revolution, the radius of the base being 4 and the altitude 10.

25. Find the lateral area, total area, and volume, of a frustum of a cone of revolution, the radii of the bases being 7 and 2 and the altitude 3.

26. Find the lateral area of a frustum of a cone of revolution, the radii of the bases being 21 and 6 inches and the altitude 36 inches. •

Ans. 3308.1 cu. ins.

27. Find the volume of a frustum of a cone of revolution, the radii of the bases being 4 and 2 feet and the altitude 9 feet.

Ans. 65.97 cu. ft.

28. The slant height of a cone of revolution is 4 feet : how far from the vertex must the slant height be cut by a plane parallel to the base that the lateral area may be divided into two equivalent parts?

29. The altitude of a cone of revolution is equal to the diameter of its base: find the ratio of the area of the base to the lateral area.

30. Find the volume of an equilateral cylinder in terms of its total area.

See Ex. (10).

31. The lateral area of a cylinder of revolution is $116\frac{2}{3}$ sq. ft., and the altitude is 14 feet: find the diameter of its base. *Ans.* 2.65 ft.

32. The volume of a cylinder is 15.7 cu. ft., and the diameter of its base is 2 feet: find its altitude. *Ans.* 5 feet.

33. The lateral area of a cylinder of revolution is 70π and its volume is 175π : find its height and the radius of its base.

34. The lateral area of a cone of revolution is 60π and its slant height is 12: find its volume.

35. Find the total area of a frustum of a cone of revolution, the radii of its bases being 9 and 4 feet and its height 12 feet. *Ans.* 266π .

36. The volume of a frustum of a cone of revolution is 920 cu. ft., and its height is 12 feet: find the radii of its bases if their sum is 8 feet.

37. Find the number of cubic feet in a log 12 feet long and $6\frac{3}{4}$ feet in diameter. *Ans.* 418.88 cu. ft.

38. Find the number of cubic feet in the trunk of a tree, 70 feet long, the diameters of its ends being 10 and 7 feet.

39. How many square inches of sheet-iron does it take to make a joint of 6-inch stovepipe $2\frac{1}{2}$ feet long, allowing an inch and a half for the seam?

40. The heights of two cylinders of revolution of equal volumes are as 9 : 16; the diameter of one of them is 6 feet: find the diameter of the other.

41. A cylinder of revolution whose base is 5 feet in diameter and a cone of revolution whose base is 6 feet in diameter, have equal volumes; the height of the cone is 10 feet: find the height of the cylinder.

42. The height of a frustum of a cone of revolution is 6 feet, and the diameters of its bases are 3 and 2 feet: find the height of a cylinder of revolution of the same volume as the frustum, and whose base is equal to the mid-section of the frustum.

See (777).

43. The volumes of two equilateral cylinders are to each other as 3 : 4 : find the ratio of their heights.

44. The volumes of two similar cones are 27 cubic feet and 216 cubic feet, and the height of the first is 9 feet : find the height of the other.

45. The volumes of two similar cones of revolution are to each other as 512 : 729 : find the ratio of their lateral areas.

46. The slant heights of two similar cones of revolution are to each other as 3 : 5 : find the ratio of their lateral areas, and of their volumes.

47. The height of a frustum of a cone is $\frac{3}{4}$ the height of the complete cone : find the ratio of the volume of the frustum to that of the cone.

48. The total areas of two similar cylinders of revolution are to each other as 25 : 49 : find the ratio of their volumes.

49. The altitude of a cone of revolution is 10, and its slant height is 14 : find the total area of the inscribed cylinder whose altitude is 6.

50. The volume of a cone of revolution is 392, and its slant height is to the diameter of its base as 100 : 56 : find its altitude and the diameter of its base.

51. Find the surface and volume of a sphere whose diameter is (1) 16 inches; and (2) 17 inches.

Ans. (1) $804\frac{1}{2}$; $2144\frac{2}{3}$; (2) 908; 2572.45.

52. Find the diameter of a sphere if the surface is (1) 1809 square inches; (2) 616 square inches; and (3) 9856 square inches.

53. The volume of a sphere is 113 : find its diameter and its surface.

54. The volume of a sphere is 776π : find its diameter and its surface.

55. The surface of a sphere is 784π : find its radius and its volume.

56. The volume of a sphere is 268.08 cubic inches: find the altitude of the circumscribing cylinder.

Ans. 8 inches.

57. The surface of a given sphere has the same numerical value as the circumference of a great circle: find the numerical value of its radius.

58. Find the surface of a zone of one base, its altitude being 10 feet, and the diameter of the sphere 100 feet.

59. Find the surface of a zone, its altitude being 6 feet, and the diameter of the sphere 20 feet.

60. Find the height of a zone whose area is equal to that of a great circle in a sphere of radius r .

61. Assuming the earth to be a sphere with a radius of 4000 miles, and the altitudes of the torrid and the temperate zones to be 3200 and 2052 miles respectively, find the areas of these two zones.

62. The surface and volume of a given sphere are expressed by the same number: find its diameter.

63. Find the weight of an iron shell 4 inches in diameter, the iron being 1 inch thick, and weighing $\frac{1}{4}$ of a pound to the cubic inch.

Ans. $7\frac{1}{8}$ lbs.

64. A sphere of radius r is cut by a plane so that the area of the greater zone is a mean proportional between the area of the smaller zone and the area of the sphere.

65. Find the surface of a sphere inscribed in a cube whose surface is 216.

66. Find the volume of a sphere circumscribed about a cube whose volume is 64.

67. If an iron ball 8 inches in diameter weighs 72 pounds, find the weight of an iron shell 10 inches in diameter, the iron being 2 inches thick.

68. A cone of revolution, the radius of whose base is 12, is inscribed in a sphere of radius 20: find the volume of the cone.

69. How high above the earth must a person be raised in order that he may see one-fifth of its surface?

70. How much of the earth's surface would a man see if he were raised to the height of the diameter above it?

71. Find the volume of a spherical sector, if the altitude of the zone which forms its base is 2 feet, and the radius of the sphere is 5 feet.

72. Find the volume of a spherical sector, if the area of the zone which forms its base is 5 square feet, and the radius of the sphere is 2 feet.

73. Find the volume of the spherical sector generated by revolving a circular sector about an axis in its plane perpendicular to one of its limiting radii, the radius of the circular sector being 6 and its central angle 30° .

74. Find the volume of a triangular spherical pyramid, if the angles of the spherical triangle which forms its base are each 120° , and the radius of the sphere is 10 feet.

75. Find the volume of a quadrangular spherical pyramid, if the planes of the four faces of the pyramid make with each other angles of 80° , 100° , 120° , 150° , and a lateral edge of the pyramid is 4 feet.

76. Find the volume of a spherical segment, the radii of whose bases are 3 and 5, and whose altitude is 4.

77. Find the volume of a spherical segment, the diameters of whose bases are 24 and 20 inches, and whose altitude is 4 inches.

78. Find the volume of a spherical segment of one base whose altitude is 4 feet, the radius of the sphere being 10 feet.

79. Find the volume of a spherical segment of one base whose altitude is 6 inches and the diameter of its base 16 inches.

80. A sphere, 2 feet in diameter, is cut by two parallel planes, one at 3 and the other at 9 inches from the centre: find the volume of the segment included between them.

81. Find the volume of a spherical segment of one base whose altitude is 20 feet and the diameter of its base 60 feet.

82. The radius of a sphere is 6 inches: find the area of a spherical triangle whose angles are 95° , 100° , 120° .

83. The radius of a sphere is 7 feet: find (1) the area of a lune whose angle is 30° , and (2) the volume of a wedge whose angle is 36° .

84. A sphere 6 inches in diameter has a hole bored through its centre with a 3-inch auger: find the remaining volume.

85. A boiler is a cylinder with hemispherical ends; its total length is 20 feet and circumference 11 feet: find its surface and the quantity of water required to fill it half full.

86. A cone is circumscribed about a sphere, and its height is double the diameter of the sphere: show that the total surface and the volume of the cone are respectively double those of the sphere.

87. A circle, radius r , revolves about a line in its plane and at a distance d from its centre: find the volume of the ring which the circle generates.



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